A noncommutative Amir-Cambern Theorem

Jean ROYDOR (RIMS Kyoto University)

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Amir-Cambern Theorem

Th. (Banach-Stone'37) :

Let $T : C(K_1) \to C(K_2)$ surjective linear isometry. Then $\exists \; u \in C(K_2, \mathbb{T})$ and $\tau : K_2 \to K_1$ homeomorphism s.t.

$$T(f) = u(f \circ \tau), \quad \forall f \in C(K_1)$$

In particular, $C(K_1) = C(K_2)$ $*$-isomorphically.

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Def. (Banach-Mazur distance):

Let $\mathcal{X}, \mathcal{Y}$ Banach spaces,

$$d(\mathcal{X}, \mathcal{Y}) = \inf\{\|T\|\|T^{-1}\| \ : \ T : \mathcal{X} \to \mathcal{Y} \text{ linear isom.}\}$$

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Th. (Amir-Cambern'66) :

If $d(C(K_1), C(K_2)) < 2$, then $C(K_1) = C(K_2)$ $*$-isomorphically.

A noncommutative analogue

$C(K)$-spaces	$\rightsquigarrow$	unital C^*-alg.
L_∞-spaces	$\rightsquigarrow$	von Neumann alg.
Uniform alg.	$\rightsquigarrow$	unital nonselfadjoint op. alg.
Banach spaces	$\rightsquigarrow$	operator spaces.

and (very important !):

Bounded linear maps $\rightsquigarrow$ Completely bounded linear maps

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Rmk: $d_{cb}(C(K_1), C(K_2)) = d(C(K_1), C(K_2))$

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Questions :

Does there exist $\varepsilon_0 > 0$ s.t. for any unital C^*-alg. $\mathcal{A}, \mathcal{B}$:
if $d_{cb}(\mathcal{A}, \mathcal{B}) < 1 + \varepsilon_0$, then $\mathcal{A} = \mathcal{B}$ $*$-isomorphically ?
If yes, can we find explicit ε_0 ?

Main Result

Th. (R.):

There exists $\varepsilon_0 > 0$ such that for any von Neumann algebras $\mathcal{M}, \mathcal{N}$, $d_{cb}(\mathcal{M}, \mathcal{N}) < 1 + \varepsilon_0$ implies $\mathcal{M} = \mathcal{N}$ $*$-isomorphically.

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- Step 1 Prove that the unitization of a cb-isomorphism with small bound is almost multiplicative.
- Step 2 Show that a vN alg. is stable under perturbations by cb-close multiplications.

Almost multiplicativity: the unital case

Th. (D. Blecher'01):

Let $T : \mathcal{A} \to \mathcal{B}$ be surjective linear complete isometry between unital operator algebras.
Then $\exists$ unitary $u \in \mathcal{B} \cap \mathcal{B}^\star$ and unital completely isometric algebra homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ s.t. $T(x) = u\pi(x)$.
In particular, if $T(1) = 1$, then T is multiplicative.

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Th. (R.):

For any $\eta > 0$, there exists $\rho \in (0, 1)$ such that for any unital operator algebras $\mathcal{A}$, $\mathcal{B}$, for any unital cb-isomorphism $T : \mathcal{A} \to \mathcal{B}$, $\|T\|_{cb} \leq 1 + \rho$ and $\|T^{-1}\|_{cb} \leq 1 + \rho$ imply $\|T^\vee\|_{cb} < \eta$.

Characterizing invertible elements

We need to prove an operator space characterization of invertible elements in a C^*-algebra.

Lemma :

Let $\mathcal{A}$ be a unital C^*-algebra and $x \in \mathcal{A}$, $\|x\| \leq 1$. Then, x is invertible if and only if there exists $\alpha > 0$ such that for any $y \in \mathcal{A}^{**}$ of norm one,

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2 \geq \alpha + \|y\|^2 \quad \text{and} \quad \left\| \begin{bmatrix} x & y \end{bmatrix} \right\|^2 \geq \alpha + \|y\|^2 \quad (C)$$

In this case, the supremum of the α's satisfying (C) equals $\|x^{-1}\|^{-2}$ and moreover, condition (C) is actually satisfied for any $y \in \mathcal{A}^{**}$.

Rmk: the 'only if' part is true in any unital operator algebra.

Almost multiplicativity: the general case

Prop.:

Let $\mathcal{A}$ be a unital operator algebra and $\mathcal{B}$ be unital C^*-algebra. Let $T : \mathcal{A} \to \mathcal{B}$ be a cb-isomorphism such that $\|T\|_{cb}\|T^{-1}\|_{cb} \leq 1+\epsilon$, with $\epsilon < \sqrt{2}-1$. Then $T(1)$ is invertible and

$$\|T(1)^{-1}\| \leq \frac{1}{\|T\|_{cb}}\sqrt{\frac{(1+\epsilon)^2}{2-(1+\epsilon)^2}}.$$

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Corollary 1:

For any $\eta > 0$, there exists $\epsilon \in (0, \sqrt{2}-1)$ such that for any unital C^*-algebras $\mathcal{A}$, $\mathcal{B}$, for any cb-isomorphism $T : \mathcal{A} \to \mathcal{B}$, $\|T\|_{cb} = 1$ and $\|T^{-1}\|_{cb} \leq 1+\epsilon$ implies $\|L^\vee\|_{cb} < \eta$, where $L = T(1)^{-1}T$.

Stability under perturbation by cb-close multiplications

Notation:
$H^k(\mathcal{A},\mathcal{A})$ the kth Hochschild cohomology group of $\mathcal{A}$ over itself.
$\mathrm{m}_{\mathcal{A}}$ denotes the original multiplication on $\mathcal{A}$.

Th. (B.E. Johnson'77, I. Raeburn & J. Taylor'77)

Let $\mathcal{A}$ be a Banach algebra satisfying

$$H^2(\mathcal{A},\mathcal{A}) = H^3(\mathcal{A},\mathcal{A}) = 0.$$

Then there exist $\delta, C > 0$ such that for every multiplication m on $\mathcal{A}$ satisfying $\| \mathrm{m} - \mathrm{m}_{\mathcal{A}} \| \leq \delta$, there is a bounded linear isomorphism $\Phi : \mathcal{A} \to \mathcal{A}$ such that

$$\|\Phi - \mathrm{id}_{\mathcal{A}}\| \leq C\| \mathrm{m} - \mathrm{m}_{\mathcal{A}} \| \quad \text{and} \quad \Phi(\mathrm{m}(x,y)) = \Phi(x)\Phi(y).$$

Hochschild cohomology

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach $\mathcal{A}$-module. Denote
$\mathcal{L}^0(\mathcal{A}, \mathcal{X}) = \mathcal{X}$ and
$\mathcal{L}^k(\mathcal{A}, \mathcal{X})$ the space of all bounded k-linear maps from $\mathcal{A}^k \to \mathcal{X}$.

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Define the 'coboundary maps' $\delta^k : \mathcal{L}^k(\mathcal{A}, \mathcal{X}) \to \mathcal{L}^{k+1}(\mathcal{A}, \mathcal{X})$ by:

$$\begin{aligned}
\delta^0(x)(a) &= ax - xa \quad \text{and} \\
\delta^k(\varphi)(a_1, ..., a_{k+1}) &= a_1\varphi(a_2, ..., a_{k+1}) \\
&\quad + \sum_{i=1}^{k-1}(-1)^i\varphi(a_1, ..., a_{i-1}, (a_i a_{i+1}), ..., a_{k+1}) \\
&\quad + (-1)^k\varphi(a_1, ..., a_k)a_{k+1}
\end{aligned}$$

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Def. :

Elements of $\operatorname{Ran} \delta^{k-1}$ are called *coboundaries*.
Elements of $\operatorname{Ker} \delta^k$ are called *cocycles*.
The kth Hochschild cohomology group is denoted:
$H^k(\mathcal{A}, \mathcal{X}) = \operatorname{Ker} \delta^k / \operatorname{Ran} \delta^{k-1}$.

Stability under perturbation by cb-close multiplications

Prop.:

Let $\mathcal{A}$ be an operator algebra satisfying

$$H^2_{cb}(\mathcal{A},\mathcal{A}) = H^3_{cb}(\mathcal{A},\mathcal{A}) = 0. \quad (\star)$$

Then there exist $\delta, C > 0$ such that for every multiplication m on $\mathcal{A}$ satisfying $\| \mathrm{m} - \mathrm{m}_{\mathcal{A}} \|_{cb} \leq \delta$, there is a completely bounded linear isomorphism $\Phi : \mathcal{A} \to \mathcal{A}$ such that

$$\|\Phi - \mathrm{id}_{\mathcal{A}}\|_{cb} \leq C\| \mathrm{m} - \mathrm{m}_{\mathcal{A}} \|_{cb} \ \text{ and } \ \Phi(\mathrm{m}(x,y)) = \Phi(x)\Phi(y).$$

Moreover, if $\mathcal{A}$ is a von Neumann algebra, then $(\star)$ is necessarily satisfied and one can choose $\delta = 2^{-1}10^{-10}$ and $C = 4$.

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Th. (E. Christensen, A. Sinclair '89):

Let $\mathcal{M}$ be a vN alg. Then, $H^k_{cb}(\mathcal{M},\mathcal{M}) = 0$ for any k.

Proof of the main result

Corollary 2:

Let $\mathcal{M}$ be a von Neumann algebra. Then for every multiplication m on $\mathcal{M}$ satisfying $\| \mathrm{m} - \mathrm{m}_{\mathcal{M}} \|_{cb} \leq 2^{-1}10^{-10}$, there is a completely bounded linear isomorphism $\Phi : \mathcal{M} \to \mathcal{M}$ such that

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