Do pseudospectra determine norm behavior of matrices with simple eigenvalues?

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Introduction

Let A be a unital Banach algebra, and let $a \in A$.
We write $\sigma(a)$ for the spectrum of a.

For $\epsilon > 0$, the ϵ-**pseudospectrum** of a is

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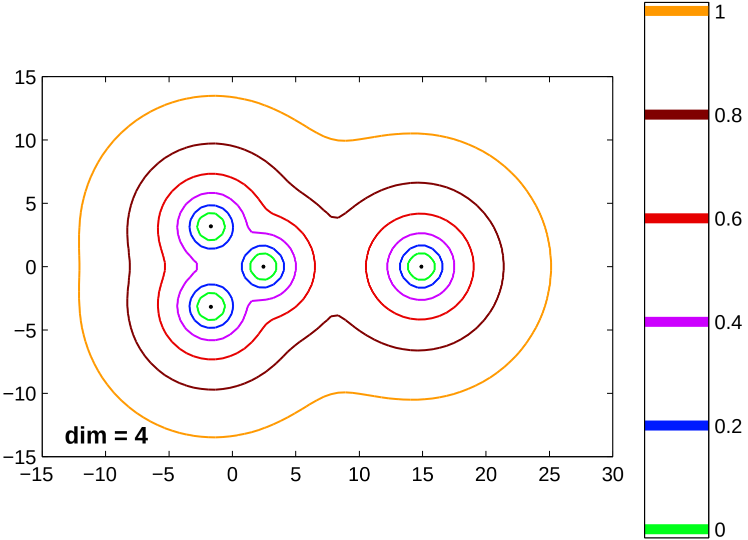
In the case where A is the C*-algebra of $n \times n$ matrices, there is a third characterization, in terms of the smallest singular value :

$$\sigma_\epsilon(a) = \{\lambda \in \mathbb{C} : s_{\min}(a - \lambda 1) < \epsilon\}.$$

Example : pseudospectra of a 4 × 4 matrix



Just how much do pseudospectra tell us ?

Naive question

Let a, b be $n \times n$ matrices such that $\sigma_\epsilon(a) = \sigma_\epsilon(b)$ for all $\epsilon > 0$.
Must a and b be unitarily equivalent ?

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Answer : No. For example, consider

$$a := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad b := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, for all $\lambda \in \mathbb{C}$,

$$\|(a - \lambda 1)^{-1}\| = \|(b - \lambda 1)^{-1}\| = \max\{|\lambda|^{-1}, |1 - \lambda|^{-1}\}.$$

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Answer : No (Greenbaum–Trefethen, 1993).

Idea : Take $a := a' \oplus c$ and $b := b' \oplus c$, where

$$\|(c - \lambda 1)^{-1}\| \geq \max\Big\{\|(a' - \lambda 1)^{-1}\|, \|(b' - \lambda 1)^{-1}\|\Big\} \quad (\lambda \in \mathbb{C}).$$

This is possible, and with enough flexibility to have $\|a\| \neq \|b\|$. However, a and b are necessarily derogatory.

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New question

Does the answer change if a, b have only simple eigenvalues ?

The answer is still 'no'

Example (Ransford–Rostand, 2011)

Let

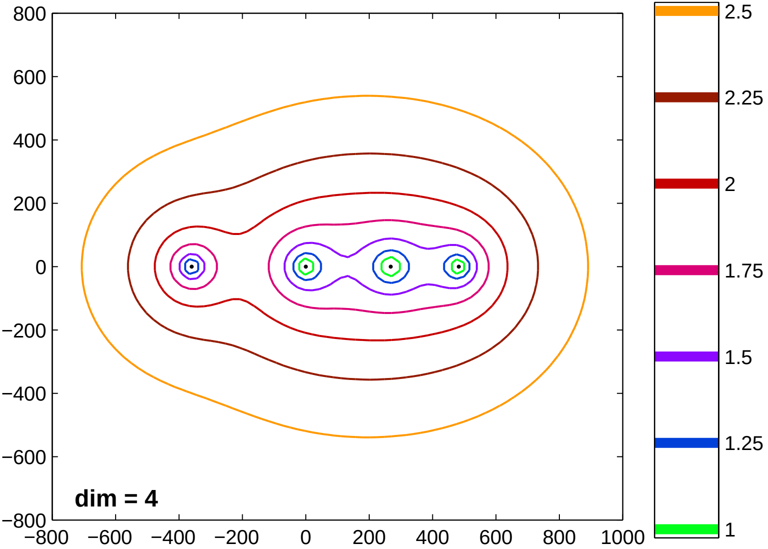
$$a := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 180 & -360 & 0 & 0 \\ -90+120\sqrt{5} & 180+60\sqrt{5} & 120\sqrt{5} & 0 \\ 450 & -180 & -360 & 216\sqrt{5} \end{pmatrix}$$

and

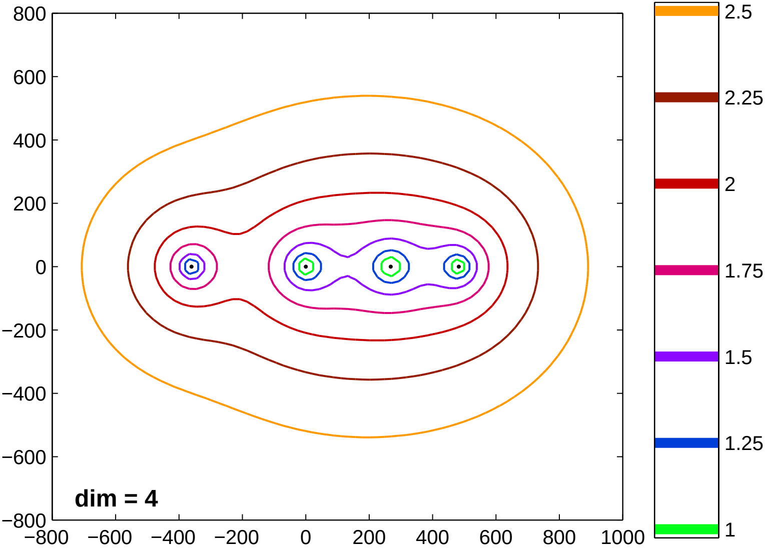
$$b := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 120 & -360 & 0 & 0 \\ 45\sqrt{130}-15\sqrt{26} & 45\sqrt{26}+15\sqrt{130} & 120\sqrt{5} & 0 \\ 30\sqrt{130} & 10\sqrt{130} & 80\sqrt{5} & 216\sqrt{5} \end{pmatrix}.$$

Then $\sigma_\epsilon(a) = \sigma_\epsilon(b)$ *for all* $\epsilon > 0$*, but* $\|a^2\| \neq \|b^2\|$.

Pseudospectra of *a*



Pseudospectra of b



Proofs

Proof that $\sigma_\epsilon(a) = \sigma_\epsilon(b)$ for all $\epsilon > 0$.
By explicit computation, for all $\lambda, \zeta \in \mathbb{C}$ we have

$$\det\Big((a - \lambda 1)(a - \lambda 1)^* - \zeta 1\Big) = \det\Big((b - \lambda 1)(b - \lambda 1)^* - \zeta 1\Big).$$

So $a - \lambda 1$ and $b - \lambda 1$ have the same singular values for all $\lambda \in \mathbb{C}$.
In particular, $s_{\min}(a - \lambda 1) = s_{\min}(b - \lambda 1)$ for all $\lambda \in \mathbb{C}$.

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Proof that $\|a^2\| \neq \|b^2\|$.
By explicit computation again,

$$\det(a^2 a^{*2} - \zeta 1) - \det(b^2 b^{*2} - \zeta 1) = \gamma \zeta^2,$$

where γ is a non-zero constant. Hence a^2, b^2 have no common singular values other than zero. In particular $\|a^2\| \neq \|b^2\|$.

How a and b were found

Lemma

Let a and b be 4×4 matrices. Then $a - \lambda 1$ and $b - \lambda 1$ have the same singular values for all $\lambda \in \mathbb{C}$ iff

(i) $\sigma(a) = \sigma(b)$,

(ii) $\mathrm{tr}(a^j a^{*k}) = \mathrm{tr}(b^j b^{*k})$ $(1 \leq j \leq k \leq 3)$, *and*

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- Note that (ii) holds iff $p \circ p^{-t} = q \circ q^{-t}$ (Hadamard product), where $p := vv^*$ and $q := ww^*$. This is independent of d.
- Fix positive matrices p, q so that $p \circ p^{-t} = q \circ q^{-t}$, and then, *if possible*, choose d so that (iii) holds for a, b but not a^2, b^2.

How a and b were found (continued)

Appropriate choice of p, q satisfying $p \circ p^{-t} = q \circ q^{-t}$:

$$p := \begin{pmatrix} 1 & * & * & 0 \\ * & 1 & 0 & * \\ \alpha & 0 & 1 & * \\ 0 & \beta & * & 1 \end{pmatrix} \quad \text{and} \quad q := \begin{pmatrix} 1 & * & * & * \\ * & 1 & 0 & 0 \\ * & 0 & 1 & * \\ * & 0 & \gamma & 1 \end{pmatrix}.$$

Here α, β, γ are free parameters, and all the other entries $*$ are determined by them.

Closing remarks

- Same construction works with the operator norm replaced by any Schatten p-norm for $p \neq 2$. For $p = 2$, pseudospectra DO determine norm behavior (Greenbaum–Trefethen, 1993).

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