# Arens regularity of the Fourier algebra

Joint work with Matthias Neufang

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# Table of contents

Background

Ultra weak amenability

Review

Strong Topological Centre

Complete description

Arens regularity

# Background

# Banach algebra

Definition

*Let $A$ be a commutative Banach algebra. A* **multiplier** *$T$ of $A$ is a bounded linear operator $T : A \to A$ such that*

$$T(ab) = T(a)b = aT(b) \qquad (a, b \in A).$$

*The set of all multipliers of $A$ is denoted by $M(A)$.*

# Second dual

**Arens Products**

for $m, n \in A^{**}$, $f \in A^*$, $a, b \in A$

**Left Arens product**: $(A^{**}, \Box)$

$$\begin{aligned} \langle m \Box n, f \rangle &= \langle m, n \Box f \rangle \\ \langle n \Box f, a \rangle &= \langle n, f \Box a \rangle \\ \langle f \Box a, b \rangle &= \langle f, ab \rangle \end{aligned}$$

# Second dual

**Arens Products**

for $m, n \in A^{**}$, $f \in A^{*}$, $a, b \in A$

**Right Arens product**: $(A^{**}, \triangle)$

$$\begin{aligned} \langle m \triangle n, f \rangle &= \langle n, f \triangle m \rangle \\ \langle f \triangle m, a \rangle &= \langle m, a \triangle f \rangle \\ \langle a \triangle f, b \rangle &= \langle f, ba \rangle \end{aligned}$$

# Second dual

## Definition

*The* **left / right topological centre**, *is defined by*

$$Z_\ell(A^{**}) = \{m \in A^{**} \mid m\Box n = m\triangle n, \forall n \in A^{**}\}$$
$$= \{m \in A^{**} \mid \lambda_m \text{ is } w^*\text{-}w^*\text{-continuous}\};$$

$$Z_r(A^{**}) = \{m \in A^{**} \mid n\Box m = n\triangle m, \forall n \in A^{**}\}$$
$$= \{m \in A^{**} \mid \rho^m \text{ is } w^*\text{-}w^*\text{-continuous}\}.$$

# Second dual

Definition

*A Banach algebra $A$ is called:*

1. **Arens regular** *if* $Z_\ell(A^{**}) = Z_r(A^{**}) = A^{**}$;
2. **Left Arens irregular** *if* $Z_\ell(A^{**}) \neq A^{**}$;
3. **Left strongly Arens irregular** *if* $Z_\ell(A^{**}) = A$.

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## Example

- $(l^1, \cdot)$, $C^*$-algebras are Arens regular.
- **(Losert, 2004)** $A(SU(3))$ is Arens irregular, but not strongly Arens irregular.
- **(Lau, L., 1986)** $(L^1(G), *)$ is left and right strongly Arens irregular.

# Fourier algebra

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### Definition

*Let $G$ be a locally compact group and $\mu$ be a left Haar measure on $G$.*

1. *We define the group algebra, $L^1(G)$, with the image of $L^2(G) \otimes_\gamma L^2(G)$ under the map $f \otimes g \to f \bullet \bar{g}$; this becomes a Banach algebra with the convolution as a product.*

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2. *We define the Fourier algebra, $A(G)$, with the image of $L^2(G) \otimes_\gamma L^2(G)$ under the map $f \otimes g \to f * \check{g}$; this becomes Banach algebra with the pointwise product.*

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# Ultra weak amenability
# of
# locally compact group

# Weak amenability

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### Example

- Amenable group.
- **(Haagerup, 1979)** $\mathbb{F}_n$, the free group over $n$ generators.
- **(Haagerup)** Subgroup of weak amenable group.

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### Example

- Weak amenable group, in particular amenable group.
- Subgroup of ultra weak amenable group.

# Review

# Conjecture

Let $G$ be a locally compact group. If $A(G)$ is Arens regular, then $G$ is finite.

# Results

- **(Lau, 1981)** Let $G$ be an amenable group. Then, the Arens regularity of $A(G)$ implies that $G$ is finite.

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- **(Lau, 1981)** Let $G$ be an amenable group. Then, the Arens regularity of $A(G)$ implies that $G$ is finite.

- **(Forrest, 1991)** If $A(G)$ is Arens regular, then $G$ is a discrete group.

# Strong Topological Centre

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Definition

**(Neufang, P.)** *The left strong topological centre of $A^{**}$ is defined by*

$$SZ_\ell(A^{**}) = \{m \in A^{**} : \lambda_m = T^{**},\ \textit{for some } T \in B(A)\}.$$

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$$\lambda_m : A^{**} \longrightarrow A^{**}$$

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**Situation:**

$$A \subseteq SZ_\ell(A^{**}) \subseteq Z_\ell(A^{**}) \subseteq A^{**}$$

# Strong Topological Centre

Theorem

**(Hu, N., R., 2010)** *Let A be a Banach algebra.*

$$SZ_\ell(A^{**}) = Z_\ell(A^{**}) \cap \{m \in A^{**} \mid m \Box A \subseteq A\}.$$

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**Corollary**

1. *If $A$ is Arens regular, then $SZ_\ell(A^{**}) = \{m \in A^{**} \mid m\Box A \subseteq A\}$. In particular for $C^*$-algebras.*
2. *If $A$ is a left ideal in $A^{**}$, then $SZ_\ell(A^{**}) = Z_\ell(A^{**})$. In particular, if $G$ is discrete, then $SZ_\ell(A(G)^{**}) = Z_\ell(A(G)^{**})$.*

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$$SZ_\ell(A^{**}) = Z_\ell(A^{**}) \cap LM(A).$$

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**(Hu, N., R., 2010)** *Let G be a locally compact amenable group.*

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Note that $Z_\ell(A(SU(3))^{**}) \neq A(SU(3))$ but $SU(3)$ **is compact**.

# Complete description of the topological centre

## Key Lemma (1)

**(Neufang, P.)** *Let $A$ be a faithful Banach algebra. Suppose that $A$ has the following property: for any $T \in LM(A)$, we have $T \in A$ if $T^*(A^*) \subseteq \overline{< A^*A >}$. Then,*

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$$SZ_\ell(A^{**}) = A \oplus \overline{< A^{**}A^* >}^\perp.$$

## Corollary

**(Neufang, P.)** *Let* $A$ *be a weakly sequentially complete Banach algebra with a sequential approximate identity. Then*

$$SZ_\ell(A^{**}) = A \oplus \overline{< A^{**}A^* >}^\perp.$$

## Key Lemma (2)

**(Neufang, P.)** *Let $A$ be a Banach algebra. Suppose there is a closed ideal $I$ in $A$ such that there is $A$-bimodule projection $p : A \to I$. Then*

$$p^{**}(SZ_\ell(A^{**})) \subseteq SZ_\ell(I^{**}).$$

### Theorem

**(Neufang, P.)** *Let G be an ultra weakly amenable group. Then*

$$SZ_\ell(A(G)^{**}) = A(G) \oplus \overline{< A(G)^{**} A(G)^{*} >}^{\perp}.$$

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- $SZ_\ell(A(G)^{**}) \subseteq Z_\ell(A(G)^{**})$
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  **(Hu 2006)** *Let $G$ be a locally compact group and let $m \in Z_\ell(A(G)^{**})$. Then $m \in A(G) \oplus \overline{< A(G)^{**}A(G)^{*} >}^{\perp}$ iff $(p_H)^{**}(m) \in A(H) \oplus \overline{< A(H)^{**}A(H)^{*} >}^{\perp}$ for all $\sigma$-compact subgroup $H$ of $G$.*

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- Key Lemma (2)

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  Let $H$ be a $\sigma$-compact open subgroup of $G$. Then $A(H)$ is a closed ideal in $A(G)$. The $A$-bimodule projection needed is

$$\begin{array}{rcl} p_H : A(G) & \longrightarrow & A(H) \\ f & \longmapsto & f \cdot 1_H \end{array}$$

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  For all $\sigma$-subgroup $H$ of $G$,

  $$SZ_\ell(A(H)^{**}) = A(H) \oplus \overline{< A(H)^{**}A(H)^* >}^{\perp}$$

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### Corollary

**(Neufang, P.)** *Let $G$ be an ultra weakly amenable discrete group. Then $A(G)$ is strongly Arens irregular if and only if*

$$\overline{< A(G)^{**}A(G)^{*} >} = A(G)^{*}.$$

V. Losert proved that $< A(G)^{**}A(G)^* >$ is not norm dense for any discrete group G containing the free group $\mathbb{F}_r$ on $r$ generators, $(2 \leq r < \infty)$

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# Arens regularity

# Main theorem

Theorem

**(Neufang, P.)** *Let $G$ be a locally compact ultra weakly amenable group. If $A(G)$ is Arens regular, then $G$ is finite.*

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- Theorem
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- Theorem
  **(Ülger 1999)** *If $A(G)$ is Arens regular, then*
  $$\overline{< A(G)^{**}A(G)^{*} >} = UCB(\hat{G})$$

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- $A(G)^{**} = Z_\ell(A(G)^{**}) = A(G) \oplus UCB(\hat{G})^\perp$

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- Lemma
  **(Bekka, K., L., S., 1998)** *If $G$ is a discrete group and $A(G)$ is $w^*$-closed in $B(G)$, then $G$ is finite.*

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- $B_\rho(G)$ is the closure of $B(G) \cap C_{00}(G)$ in the w$^*$-topology of $B(G)$, so $A(G)$ is weak$^*$ closed in $B(G)$.
- Lemma
  **(Bekka, K., L., S., 1998)** *If $G$ is a discrete group and $A(G)$ is $w^*$-closed in $B(G)$, then $G$ is finite.*
- With $A(G) = B_\rho(G)$, we conclude that $G$ is finite.

### Corollary

**(Neufang, P.)** *Let $G$ be a locally compact infinite group with an infinite ultra weakly amenable subgroup. Then $A(G)$ is not Arens regular.*

# Question

Let $G$ be a $\sigma$-compact (discrete) group such that $A(G)$ has an approximate identity. Is $A(G)$ have a sequential approximate identity ?

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A positive answer to this question will give the following:

**Let $G$ be a $\sigma$-compact (discrete) group such that $A(G)$ has an approximate identity. If $A(G)$ is Arens regular, then $G$ is finite.**

# Approach for Tarski's monster groups

Olshanskii proved that there is a non-amenable group $G$ such that $\mathbb{F}_n$ is not a subgroup of $G$. In fact, this group is such that any proper subgroup is finite. Such exotic groups are called Tarski's monster groups.

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With the older approaches, we can say that if $A(G)$ is Arens regular, then $G$ has no infinite amenable subgroup. So it could be possible that for a Tarski's monster group $G$, $A(G)$ is Arens regular.

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With our approach, if $A(G)$ is Arens regular, then $G$ has no infinite ultra weakly amenable subgroup. But still, for a Tarski's monster group $G$, $A(G)$ can be Arens regular.

# Question

Are Tarski's monster groups weakly amenable or ultra weakly amenable ?

# How answer this question

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Olshanskii, Osin and Sapir introduced a new class of groups: Lacunary hyperbolic groups.

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Olshanskii, Osin and Sapir introduced a new class of groups: Lacunary hyperbolic groups.

Example

- Torsion group.
- Hyperbolic group.
- Tarski's monster group.

# Characterization of Lacunary hyperbolic groups

Let $G$ be a lacunary hyperbolic group. Then $G$ is the direct limit of a sequence of finitely generated groups and epimorphisms

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \dots$$

such that $G_i$ is generated by a finite set $S_i$, $\alpha_i(S_i) = S_{i+1}$, and each $G_i$ is hyperbolic.

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Ozawa proved that hyperbolic groups are weakly amenable.