$(DN) - (\Omega)$ type conditions for Fréchet operator spaces

Krzysztof Piszczek

Adam Mickiewicz University
Poznań, Poland
e-mail: kpk@amu.edu.pl

Banach Algebras 2011,
Waterloo, Canada, 3-10 August, 2011.

A Fréchet space X is a metrizable and complete lcs.

A Fréchet space X is a metrizable and complete lcs.

$$X = \left(X, (||\cdot||_n)_{n\in\mathbb{N}}\right)$$

A Fréchet space X is a metrizable and complete lcs.

$$X = (X, (||\cdot||_n)_{n\in\mathbb{N}}) \quad \text{or} \quad X = \mathrm{proj}_k(X_k, \iota_k^{k+1}),$$

where

$$X_k = \left(X/_{\mathrm{Ker}||\cdot||_k}, ||\cdot||_k\right)^{\sim}, \qquad \iota_k^{k+1}\colon X_{k+1} \to X_k.$$

A Fréchet space X is a metrizable and complete lcs.

$$X = (X, (||\cdot||_n)_{n\in\mathbb{N}}) \quad \text{or} \quad X = \operatorname{proj}_k(X_k, \iota_k^{k+1}),$$

where

$$X_k = \left(X\big/_{\operatorname{Ker}||\cdot||_k}, ||\cdot||_k\right)^{\sim}, \qquad \iota_k^{k+1}\colon X_{k+1} \to X_k.$$

Examples:

A Fréchet space X is a metrizable and complete lcs.

$$X = \big(X, (||\cdot||_n)_{n\in\mathbb{N}}\big) \quad \text{or} \quad X = \operatorname{proj}_k \big(X_k, \iota_k^{k+1}\big),$$

where

$$X_k = \left(X \diagup_{\mathrm{Ker}||\cdot||_k}, ||\cdot||_k\right)^{\sim}, \qquad \iota_k^{k+1} \colon X_{k+1} \to X_k.$$

Examples:

$\mathcal{S}(\mathbb{R}^n)$,

A Fréchet space X is a metrizable and complete lcs.

$$X = \big(X, (||\cdot||_n)_{n\in\mathbb{N}}\big) \quad \text{or} \quad X = \operatorname{proj}_k\big(X_k, \iota_k^{k+1}\big),$$

where

$$X_k = \big(X/_{\mathrm{Ker}||\cdot||_k}, ||\cdot||_k\big)^{\sim}, \qquad \iota_k^{k+1}\colon X_{k+1} \to X_k.$$

Examples:

$\mathcal{S}(\mathbb{R}^n)$, $\mathcal{H}(\mathbb{C}^n)$,

A Fréchet space X is a metrizable and complete lcs.

$$X = \left(X, (||\cdot||_n)_{n\in\mathbb{N}}\right) \quad \text{or} \quad X = \operatorname{proj}_k\left(X_k, \iota_k^{k+1}\right),$$

where

$$X_k = \left(X/_{\mathrm{Ker}||\cdot||_k}, ||\cdot||_k\right)^{\sim}, \qquad \iota_k^{k+1}\colon X_{k+1} \to X_k.$$

Examples:

$\mathcal{S}(\mathbb{R}^n)$, $\mathcal{H}(\mathbb{C}^n)$, $\mathcal{H}(\mathbb{D}^n)$,

A Fréchet space X is a metrizable and complete lcs.

$$X = \big(X, (||\cdot||_n)_{n\in\mathbb{N}}\big) \quad \text{or} \quad X = \mathrm{proj}_k\big(X_k, \iota_k^{k+1}\big),$$

where

$$X_k = \left(X\diagup_{\mathrm{Ker}||\cdot||_k}, ||\cdot||_k\right)^{\sim}, \qquad \iota_k^{k+1}\colon X_{k+1} \to X_k.$$

Examples:

$$\mathcal{S}(\mathbb{R}^n), \quad \mathcal{H}(\mathbb{C}^n), \quad \mathcal{H}(\mathbb{D}^n), \quad C^\infty(K),$$

A Fréchet space X is a metrizable and complete lcs.

$$X = \big(X, (||\cdot||_n)_{n\in\mathbb{N}}\big) \quad \text{or} \quad X = \operatorname{proj}_k \big(X_k, \iota_k^{k+1}\big),$$

where

$$X_k = \left(X \big/ _{\operatorname{Ker}||\cdot||_k}, ||\cdot||_k\right)^{\sim}, \qquad \iota_k^{k+1} \colon X_{k+1} \to X_k.$$

Examples:

$$\mathcal{S}(\mathbb{R}^n), \quad \mathcal{H}(\mathbb{C}^n), \quad \mathcal{H}(\mathbb{D}^n), \quad C^\infty(K), \quad C^\infty(\Omega),$$

A Fréchet space X is a metrizable and complete lcs.

$$X = \big(X, (||\cdot||_n)_{n\in\mathbb{N}}\big) \quad \text{or} \quad X = \mathrm{proj}_k \big(X_k, \iota_k^{k+1}\big),$$

where

$$X_k = \left(X\big/_{\mathrm{Ker}||\cdot||_k}, ||\cdot||_k\right)^{\sim}, \qquad \iota_k^{k+1} \colon X_{k+1} \to X_k.$$

Examples:

$$\mathcal{S}(\mathbb{R}^n), \quad \mathcal{H}(\mathbb{C}^n), \quad \mathcal{H}(\mathbb{D}^n), \quad C^\infty(K), \quad C^\infty(\Omega), \quad \omega.$$

1. Topological invariants.

1. Topological invariants.
   (i) (DN) is inherited by subspaces.

1. Topological invariants.
   (i) (DN) is inherited by subspaces.
   (ii) (Ω) passes onto quotients.

1. Topological invariants.
   (i) (DN) is inherited by subspaces.
   (ii) (Ω) passes onto quotients.
   (iii) Full characterization of subspaces, quotients and complemented subspaces of $\mathcal{S}(\mathbb{R}^n)$.

2. Splitting.

2. Splitting.

$$0 \longrightarrow X \xrightarrow{j} Y \xrightarrow{q} Z \longrightarrow 0, \quad \mathrm{Im} j = \mathrm{Ker} q, \quad q-\text{open}.$$

2. Splitting.

$$0 \longrightarrow X \xrightarrow{j} Y \xrightarrow{q} Z \longrightarrow 0, \quad \mathrm{Im} j = \mathrm{Ker} q, \quad q-\text{open}.$$

When $\quad q \circ R = \mathrm{id}_Z$?

2. Splitting.

$$0 \longrightarrow X \xrightarrow{j} Y \xrightarrow{q} Z \longrightarrow 0, \quad \mathrm{Im} j = \mathrm{Ker} q, \quad q-\text{open}.$$

When $\quad q \circ R = \mathrm{id}_Z? \quad \Leftrightarrow \quad L \circ j = \mathrm{id}_X?$

2. Splitting.

$$0 \longrightarrow X \xrightarrow{j} Y \xrightarrow{q} Z \longrightarrow 0, \quad \mathrm{Im}\, j = \mathrm{Ker}\, q, \quad q\text{-open}.$$

When $\quad q \circ R = \mathrm{id}_Z? \quad \Leftrightarrow \quad L \circ j = \mathrm{id}_X?$

Theorem (Vogt, 1980)

Suppose we are given a short exact sequence of Fréchet-Hilbert spaces

$$0 \longrightarrow X \xrightarrow{j} Y \xrightarrow{q} Z \longrightarrow 0.$$

If $X \in (\Omega)$ and $Z \in (DN)$ then the sequence splits.

3. Pełczyński's question.

Question (Pełczyński, 1970)

Does every complemented subspace of a nuclear Fréchet space with a basis has a basis itself?

3. Pełczyński's question.

Question (Pełczyński, 1970)

Does every complemented subspace of a nuclear Fréchet space with a basis has a basis itself?

Positive answers in all tame spaces.

3. Pełczyński's question.

Question (Pełczyński, 1970)

Does every complemented subspace of a nuclear Fréchet space with a basis has a basis itself?

Positive answers in all tame spaces.

Theorem (P., 2009)

If $X \in (\underline{DN}) \cap (\overline{\Omega})$ then X is tame.

$\mathcal{H}(\mathbb{D})$ with norms $||f||_n := \sup\left\{|f(z)|:\ \ |z|\leqslant 1-\dfrac{1}{n}\right\},\ n\in\mathbb{N}.$

$$\mathcal{H}(\mathbb{D}) \text{ with norms } ||f||_n := \sup\left\{|f(z)|: \ |z|\leqslant 1-\frac{1}{n}\right\}, \ n \in \mathbb{N}.$$

$$\log ||f||_m \leqslant \theta_m \log ||f||_2 + (1-\theta_m) \log ||f||_n \quad (n > m).$$

$$\mathcal{H}(\mathbb{D}) \text{ with norms } ||f||_n := \sup\left\{|f(z)|: \ |z|\leqslant 1-\frac{1}{n}\right\}, \ n\in\mathbb{N}.$$

$$\log||f||_m \leqslant \theta_m \log||f||_2 + (1-\theta_m)\log||f||_n \quad (n>m).$$

$$||f||_m \leqslant ||f||_2^{\theta_m} ||f||_n^{1-\theta_m}.$$

$$\mathcal{H}(\mathbb{D}) \text{ with norms } ||f||_n := \sup\left\{|f(z)| : \ |z| \leqslant 1 - \frac{1}{n}\right\}, \ n \in \mathbb{N}.$$

$$\log ||f||_m \leqslant \theta_m \log ||f||_2 + (1-\theta_m) \log ||f||_n \quad (n > m).$$

$$||f||_m \leqslant ||f||_2^{\theta_m} ||f||_n^{1-\theta_m}.$$

Definition

$X \in (\underline{DN})$

$$\mathcal{H}(\mathbb{D}) \text{ with norms } ||f||_n := \sup\left\{|f(z)|:\ |z|\leqslant 1-\frac{1}{n}\right\},\ n\in\mathbb{N}.$$

$$\log||f||_m \leqslant \theta_m \log||f||_2 + (1-\theta_m)\log||f||_n \quad (n>m).$$

$$||f||_m \leqslant ||f||_2^{\theta_m} ||f||_n^{1-\theta_m}.$$

Definition

$$X \in (\underline{DN}) \ \Leftrightarrow\ \exists\, p\ \forall\, q\ \exists\, r, \theta\in(0,1), C>0:$$

$$||x||_q \leqslant C||x||_p^{\theta}||x||_r^{1-\theta}\ \ \forall\, x\in X.$$

$$\mathcal{H}(\mathbb{D}) \text{ with norms } ||f||_n := \sup\left\{|f(z)|:\ \ |z|\leqslant 1-\frac{1}{n}\right\},\ \ n\in\mathbb{N}.$$

$$\log||f||_m \leqslant \theta_m \log||f||_2 + (1-\theta_m)\log||f||_n \quad (n>m).$$

$$||f||_m \leqslant ||f||_2^{\theta_m}||f||_n^{1-\theta_m}.$$

Definition

$$X\in(\underline{DN}) \ \Leftrightarrow\ \exists\, p\ \forall\, q\ \exists\, r, \theta\in(0,1), C>0:$$

$$||x||_q \leqslant C||x||_p^{\theta}||x||_r^{1-\theta}\ \ \forall\, x\in X.$$

'$\ldots\forall\,\theta\in(0,1)\ \exists\, r, C>0\ldots$' gives $X\in(DN)$.

Interpolation viewpoint.

Interpolation viewpoint.

$$X = \mathrm{proj}_k X_k,$$

Interpolation viewpoint.

$X = \mathrm{proj}_k X_k$, X_q intermediate with respect to (X_p, X_r).

Interpolation viewpoint.

$X = \mathrm{proj}_k X_k$, X_q intermediate with respect to (X_p, X_r).
(DN) type means X_q is of class $\mathcal{C}_J(\theta)$.

Interpolation viewpoint.

$X = \mathrm{proj}_k X_k$, X_q intermediate with respect to (X_p, X_r).
(DN) type means X_q is of class $\mathcal{C}_J(\theta)$.

Examples.

$\mathcal{H}(\mathbb{D}^n) \in (\underline{DN})$,

Interpolation viewpoint.

$X = \mathrm{proj}_k X_k,\;\; X_q$ intermediate with respect to (X_p, X_r).
(DN) type means X_q is of class $\mathcal{C}_J(\theta)$.

Examples.

$\mathcal{H}(\mathbb{D}^n) \in (\underline{DN}), \quad \mathcal{S}(\mathbb{R}^n), \mathcal{H}(\mathbb{C}^n) \in (DN).$

For every $\phi \in X'$ $\quad ||\phi||_k^* := \sup\{|\phi(x)| : \ ||x||_k \leqslant 1\} < +\infty \ \forall\, k \geqslant k_0.$

For every $\phi \in X'$ $\quad ||\phi||_k^* := \sup\{|\phi(x)|: \ ||x||_k \leqslant 1\} < +\infty \ \forall\, k \geqslant k_0$.
We look for

$$||\phi||_q^* \leqslant C\left(||\phi||_p^*\right)^{1-\theta}\left(||\phi||_r^*\right)^{\theta}.$$

For every $\phi \in X'$ $\quad ||\phi||_k^* := \sup\{|\phi(x)|: \; ||x||_k \leqslant 1\} < +\infty \;\; \forall\, k \geqslant k_0$.
We look for

$$||\phi||_q^* \leqslant C\left(||\phi||_p^*\right)^{1-\theta}\left(||\phi||_r^*\right)^{\theta}.$$

'$\forall\, p \;\exists\, q \;\forall\, r \;\exists \theta \in (0,1), C > 0 \ldots$' $\Rightarrow \;\; X \in (\Omega)$.

For every $\phi \in X'$ $\quad ||\phi||_k^* := \sup\{|\phi(x)|:\ ||x||_k \leqslant 1\} < +\infty\ \forall\, k \geqslant k_0$.
We look for
$$||\phi||_q^* \leqslant C\left(||\phi||_p^*\right)^{1-\theta}\left(||\phi||_r^*\right)^{\theta}.$$
'$\forall\, p\ \exists\, q\ \forall\, r\ \exists \theta \in (0,1), C > 0 \ldots$' $\Rightarrow\ X \in (\Omega)$.
'$\forall\, p, \theta \in (0,1) \ldots$' $\Rightarrow\ X \in (\overline{\Omega})$.

For every $\phi \in X'$ $\quad ||\phi||_k^* := \sup\{|\phi(x)| : \ ||x||_k \leqslant 1\} < +\infty \ \forall\, k \geqslant k_0$.
We look for

$$||\phi||_q^* \leqslant C\left(||\phi||_p^*\right)^{1-\theta}\left(||\phi||_r^*\right)^{\theta}.$$

'$\forall\, p\ \exists\, q\ \forall\, r\ \exists \theta \in (0,1), C > 0 \ldots$' $\Rightarrow \ X \in (\Omega)$.
'$\forall\, p, \theta \in (0,1) \ldots$' $\Rightarrow \ X \in (\overline{\Omega})$.

Examples.

$\mathcal{S}(\mathbb{R}^n), \mathcal{H}(\mathbb{C}^n) \in (\Omega)$,

For every $\phi \in X'$ $\quad ||\phi||_k^* := \sup\{|\phi(x)|: \ ||x||_k \leqslant 1\} < +\infty \ \forall\, k \geqslant k_0$.
We look for

$$||\phi||_q^* \leqslant C\left(||\phi||_p^*\right)^{1-\theta}\left(||\phi||_r^*\right)^{\theta}.$$

'$\forall\, p\ \exists\, q\ \forall\, r\ \exists \theta \in (0,1), C > 0 \ldots$' $\Rightarrow$ $X \in (\Omega)$.
'$\forall\, p, \theta \in (0,1) \ldots$' $\Rightarrow$ $X \in (\overline{\Omega})$.

Examples.

$\mathcal{S}(\mathbb{R}^n), \mathcal{H}(\mathbb{C}^n) \in (\Omega), \qquad \mathcal{H}(\mathbb{D}^n) \in (\overline{\Omega})$.

Definition (Effros, Webster, 1997)

X is a Fréchet operator space if $X = \mathrm{proj}_k\big(X_k, \iota_k^{k+1}\big)$,
X_k – operator space, $\iota_k^{k+1} \colon X_{k+1} \to X_k$ – completely bounded.

Definition (Effros, Webster, 1997)

X is a Fréchet operator space if $X = \operatorname{proj}_k \left(X_k, \iota_k^{k+1}\right)$,
X_k – operator space, $\iota_k^{k+1} \colon X_{k+1} \to X_k$ – completely bounded.

$$M_n(X) = \operatorname{proj}_k \Big(M_n(X_k), \left(\iota_k^{k+1}\right)_n \Big).$$

Definition (Effros, Webster, 1997)

X is a Fréchet operator space if $X = \mathrm{proj}_k\left(X_k, \iota_k^{k+1}\right)$,
X_k – operator space, $\iota_k^{k+1}\colon X_{k+1} \to X_k$ – completely bounded.

$$M_n(X) = \mathrm{proj}_k\Big(M_n(X_k), \left(\iota_k^{k+1}\right)_n\Big).$$

If $Q\colon \mathfrak{B} \to \mathfrak{O}$ – strict quantization then

$$Q(X) := \mathrm{proj}_k\Big(Q(X_k), Q\left(\iota_k^{k+1}\right)\Big).$$

$$s := \left\{ x = (x_j)_j : \ \sum_{j=1}^{+\infty} j^{2k} |x_j|^2 < +\infty \ \forall\, k \in \mathbb{N} \right\} = \mathrm{proj}_k \ell_2\big((j^k)_j\big),$$

$$s := \left\{ x = (x_j)_j : \ \sum_{j=1}^{+\infty} j^{2k} |x_j|^2 < +\infty \ \forall\, k \in \mathbb{N} \right\} = \mathrm{proj}_k \ell_2\big((j^k)_j\big),$$

$$(s)_r := \mathrm{proj}_k \ell_2\big((j^k)_j\big)_r,$$

$$(s)_c := \mathrm{proj}_k \ell_2\big((j^k)_j\big)_c,$$

$$s := \left\{ x = (x_j)_j : \ \sum_{j=1}^{+\infty} j^{2k} |x_j|^2 < +\infty \ \forall\, k \in \mathbb{N} \right\} = \operatorname{proj}_k \ell_2\big((j^k)_j\big),$$

$$(s)_r := \operatorname{proj}_k \ell_2\big((j^k)_j\big)_r,$$

$$(s)_c := \operatorname{proj}_k \ell_2\big((j^k)_j\big)_c,$$

$$\mathcal{T} := (s)_r \widehat{\otimes}_{op} (s)_c = \operatorname{proj}_k \Big(\ell_2\big((j^k)_j\big)_r \widehat{\otimes}_{op} \ell_2\big((j^k)_j\big)_c \Big).$$

Definition

$X \in (oDN)$ if (quantifiers)

$$\forall\, n \in \mathbb{N}\ \forall (x_{ij}) \in M_n(X): \quad ||(x_{ij})||_q \leqslant C ||(x_{ij})||_p^{1-\theta} ||(x_{ij})||_r^{\theta}.$$

Definition

$X \in (oDN)$ if (quantifiers)

$$\forall\, n \in \mathbb{N}\ \forall\, (x_{ij}) \in M_n(X): \ \ ||(x_{ij})||_q \leqslant C||(x_{ij})||_p^{1-\theta}||(x_{ij})||_r^{\theta}.$$

In particular, $M_n(X) \in (DN)\ \forall\, n \in \mathbb{N}$.

Definition

$X \in (oDN)$ if (quantifiers)

$$\forall\, n \in \mathbb{N}\ \forall (x_{ij}) \in M_n(X):\ \ ||(x_{ij})||_q \leqslant C ||(x_{ij})||_p^{1-\theta} ||(x_{ij})||_r^{\theta}.$$

In particular, $M_n(X) \in (DN)\ \forall\, n \in \mathbb{N}$.
If $X \in (DN)$ then by

$$||(x_{ij})|| \leqslant \Big(\sum_{i,j=1}^{n} ||x_{ij}||^2 \Big)^{\frac{1}{2}} \leqslant n ||(x_{ij})||$$

we get $M_n(X) \in (DN)\ \forall\, n \in \mathbb{N}$ (caution: C grows with n).

Definition

$X \in (o\Omega)$ if (quantifiers)

$$\forall\, n \in \mathbb{N}\ \forall (\phi_{ij}) \in M_n(X')\colon\ \ ||(\phi_{ij})||_q^* \leqslant C \left(||(\phi_{ij})||_p^*\right)^{1-\theta} \left(||(\phi_{ij})||_r^*\right)^{\theta},$$

where

$$||(\phi_{ij})||_k^* = ||(\phi_{ij})||_{CB(X_k, M_n)}.$$

Definition

$X \in (o\Omega)$ if (quantifiers)

$$\forall\, n \in \mathbb{N}\ \forall (\phi_{ij}) \in M_n(X'):\ \ ||(\phi_{ij})||_q^* \leqslant C \left(||(\phi_{ij})||_p^*\right)^{1-\theta} \left(||(\phi_{ij})||_r^*\right)^{\theta},$$

where

$$||(\phi_{ij})||_k^* = ||(\phi_{ij})||_{CB(X_k, M_n)}.$$

Proposition (P., 2011)

$(s)_r \widehat{\otimes}_{op} (s)_c$ satisfies both (oDN) and $(o\Omega)$.

If $X = \mathrm{proj}_k X_k$ then

$$MIN(X) := \mathrm{proj}_k MIN(X_k), \qquad MAX(X) := \mathrm{proj}_k MAX(X_k).$$

If $X = \text{proj}_k X_k$ then

$$MIN(X) := \text{proj}_k MIN(X_k), \qquad MAX(X) := \text{proj}_k MAX(X_k).$$

If $H = \text{proj}_k H_k$ is a Fréchet-Hilbert space then

$$H_r := \text{proj}_k (H_k)_r, \qquad H_c := \text{proj}_k (H_k)_c, \qquad OH := \text{proj}_k OH_k.$$

If $X = \mathrm{proj}_k X_k$ then

$$MIN(X) := \mathrm{proj}_k MIN(X_k), \qquad MAX(X) := \mathrm{proj}_k MAX(X_k).$$

If $H = \mathrm{proj}_k H_k$ is a Fréchet-Hilbert space then

$$H_r := \mathrm{proj}_k (H_k)_r, \qquad H_c := \mathrm{proj}_k (H_k)_c, \qquad OH := \mathrm{proj}_k OH_k.$$

Problem

Suppose $X \in (DN)$ or $X \in (\Omega)$. Which of the above quantizations carry over these conditions onto the o.s. structure?

Proposition (Vogt, 1977)

$$X \in (DN) \;\Leftrightarrow\; \exists\, p\; \forall\, q\; \exists\, r, C > 0:$$

$$U_q^\circ \subset sU_p^\circ + \frac{C}{s} U_r^\circ \qquad \forall\, s > 0,$$

Proposition (Vogt, 1977)

$$X \in (DN) \;\Leftrightarrow\; \exists\, p \,\forall\, q \,\exists\, r, C > 0:$$

$$U_q^\circ \subset sU_p^\circ + \frac{C}{s} U_r^\circ \qquad \forall\, s > 0,$$

where

$$U_k^\circ := \{\phi \in X' : \ |\phi(x)| \leqslant 1 \ \forall\, x \in U_k\}.$$

Proposition (Vogt, 1977)

$$X \in (DN) \;\; \Leftrightarrow \;\; \exists\, p \,\forall\, q \,\exists\, r, C > 0:$$

$$U_q^\circ \subset sU_p^\circ + \frac{C}{s} U_r^\circ \qquad \forall\, s > 0,$$

where

$$U_k^\circ := \{\phi \in X' : \; |\phi(x)| \leqslant 1 \;\; \forall\, x \in U_k\}.$$

Proposition (P., 2011)

$X \in (DN) \;\; \Leftrightarrow \;\; MIN(X) \in (oDN).$

Proof.

Necessity: obvious.

Sufficiency. If $(x_{ij}) \in M_n(MIN(X))$ then

$$||(x_{ij})||_k = \sup\Big\{ \big|\big|\big(\phi(x_{ij})\big)_{i,j=1}^{n}\big|\big|_{M_n} : \ \phi \in U_k^\circ \Big\}.$$

Proof.

Necessity: obvious.

Sufficiency. If $(x_{ij}) \in M_n(MIN(X))$ then

$$||(x_{ij})||_k = \sup\Big\{\Big|\Big|\big(\phi(x_{ij})\big)_{i,j=1}^n\Big|\Big|_{M_n} : \ \phi \in U_k^\circ\Big\}.$$

Consequently,

$$||(x_{ij})||_q \leqslant \sup\{||(s\xi + Cs^{-1}\eta)(x_{ij})||_{M_n} : \ \xi \in U_p^\circ, \eta \in U_r^\circ\}$$

Proof.

Necessity: obvious.
Sufficiency. If $(x_{ij}) \in M_n(MIN(X))$ then

$$||(x_{ij})||_k = \sup\Big\{ \big|\big| \big(\phi(x_{ij})\big)_{i,j=1}^n \big|\big|_{M_n} : \ \phi \in U_k^\circ \Big\}.$$

Consequently,

$$||(x_{ij})||_q \leqslant \sup\{||(s\xi + Cs^{-1}\eta)(x_{ij})||_{M_n} : \ \xi \in U_p^\circ, \eta \in U_r^\circ\} \leqslant$$

$$\leqslant s||(x_{ij})||_p + Cs^{-1}||(x_{ij})||_r.$$

Proof.

Necessity: obvious.
Sufficiency. If $(x_{ij}) \in M_n(MIN(X))$ then

$$||(x_{ij})||_k = \sup\Big\{\big|\big|\big(\phi(x_{ij})\big)_{i,j=1}^n\big|\big|_{M_n} : \ \phi \in U_k^\circ\Big\}.$$

Consequently,

$$||(x_{ij})||_q \leqslant \sup\{||(s\xi + Cs^{-1}\eta)(x_{ij})||_{M_n} : \ \xi \in U_p^\circ, \eta \in U_r^\circ\} \leqslant$$

$$\leqslant s||(x_{ij})||_p + Cs^{-1}||(x_{ij})||_r.$$

Finally,

$$||(x_{ij})||_q^2 \leqslant 4C||(x_{ij})||_p||(x_{ij})||_r.$$

□

Corollary

$X \in (\Omega) \;\Leftrightarrow\; MAX(X) \in (o\Omega).$

Corollary

$X \in (\Omega) \;\Leftrightarrow\; MAX(X) \in (o\Omega)$.

Proof.

Necessity: obvious.
Sufficiency. If $(\phi_{ij}) \in M_n(MAX(X'))$ then

$$||(\phi_{ij})||_k^* = \sup\left\{ \left|\left| \left(x''(\phi_{ij}) \right)_{i,j=1}^n \right|\right|_{M_n} : \; x'' \in U_k^{\circ\circ} \right\}$$

Corollary

$X \in (\Omega) \;\;\Leftrightarrow\;\; MAX(X) \in (o\Omega).$

Proof.

Necessity: obvious.
Sufficiency. If $(\phi_{ij}) \in M_n(MAX(X'))$ then

$$||(\phi_{ij})||_k^* = \sup\Big\{ \Big|\Big| \big(x''(\phi_{ij})\big)_{i,j=1}^n \Big|\Big|_{M_n} : \; x'' \in U_k^{\circ\circ} \Big\} =$$

$$= \sup\Big\{ \Big|\Big| \big(x''(\phi_{ij})\big) \Big|\Big|_{M_n} : \; x'' \in \overline{U_k}^{\sigma(X'',X')} \Big\}$$

Corollary

$X \in (\Omega) \;\Leftrightarrow\; MAX(X) \in (o\Omega).$

Proof.

Necessity: obvious.
Sufficiency. If $(\phi_{ij}) \in M_n(MAX(X'))$ then

$$||(\phi_{ij})||_k^* = \sup\left\{ \left|\left| \left(x''(\phi_{ij})\right)_{i,j=1}^n \right|\right|_{M_n} : \; x'' \in U_k^{\circ\circ} \right\} =$$

$$= \sup\left\{ \left|\left| \left(x''(\phi_{ij})\right) \right|\right|_{M_n} : \; x'' \in \overline{U_k}^{\sigma(X'',X')} \right\} =$$

$$\sup\left\{ \left|\left| \left(\phi_{ij}(x)\right) \right|\right|_{M_n} : \; x \in U_k \right\}.$$

$\square$

If $(x_{ij}) \in M_n(MAX(X))$ then

$$||(x_{ij})||_k = \sup \left|\left|\left(f_{uv}(x_{ij})\right)\right|\right|_{M_{nm}},$$

where the 'sup' runs over all $(f_{uv}) \in L(X, M_m)$ with $||(f_{uv})||_{L(X_k, M_m)} \leqslant 1$ and all $m \in \mathbb{N}$.

If $(x_{ij}) \in M_n(MAX(X))$ then

$$||(x_{ij})||_k = \sup \left|\left|\left(f_{uv}(x_{ij})\right)\right|\right|_{M_{nm}},$$

where the 'sup' runs over all $(f_{uv}) \in L(X, M_m)$ with $||(f_{uv})||_{L(X_k, M_m)} \leqslant 1$ and all $m \in \mathbb{N}$.
If X is Fréchet and $\dim E < +\infty$ then $L(X, E) = (X \otimes_\pi E')'$.

If $(x_{ij}) \in M_n(MAX(X))$ then

$$||(x_{ij})||_k = \sup \left|\left|\left(f_{uv}(x_{ij})\right)\right|\right|_{M_{nm}},$$

where the 'sup' runs over all $(f_{uv}) \in L(X, M_m)$ with $||(f_{uv})||_{L(X_k, M_m)} \leqslant 1$ and all $m \in \mathbb{N}$.

If X is Fréchet and $\dim E < +\infty$ then $L(X, E) = (X \otimes_\pi E')'$.

$\left(\overline{\Gamma(U_k \otimes B_{E'})}\right)_{k \in \mathbb{N}}$ – 0-nbhd basis in $X \otimes_\pi E'$,

If $(x_{ij}) \in M_n(MAX(X))$ then

$$||(x_{ij})||_k = \sup \left|\left|(f_{uv}(x_{ij}))\right|\right|_{M_{nm}},$$

where the 'sup' runs over all $(f_{uv}) \in L(X, M_m)$ with $||(f_{uv})||_{L(X_k,M_m)} \leqslant 1$ and all $m \in \mathbb{N}$.

If X is Fréchet and $\dim E < +\infty$ then $L(X, E) = (X \otimes_\pi E')'$.

$\left(\overline{\Gamma(U_k \otimes B_{E'})}\right)_{k\in\mathbb{N}}$ – 0-nbhd basis in $X \otimes_\pi E'$,

$\left(\overline{\Gamma(U_k \otimes B_{E'})}\right)^\circ = B_{L(X_k,E)}$.

If $(x_{ij}) \in M_n(MAX(X))$ then

$$||(x_{ij})||_k = \sup \left|\left|\left(f_{uv}(x_{ij})\right)\right|\right|_{M_{nm}},$$

where the 'sup' runs over all $(f_{uv}) \in L(X, M_m)$ with $||(f_{uv})||_{L(X_k, M_m)} \leqslant 1$ and all $m \in \mathbb{N}$.

If X is Fréchet and $\dim E < +\infty$ then $L(X, E) = (X \otimes_\pi E')'$.

$\left(\overline{\Gamma(U_k \otimes B_{E'})}\right)_{k \in \mathbb{N}}$ – 0-nbhd basis in $X \otimes_\pi E'$,

$\left(\overline{\Gamma(U_k \otimes B_{E'})}\right)^{\circ} = B_{L(X_k, E)}$.

If $X \otimes_\pi E' \in (DN)$ then $B_{L(X_q, E)} \subset sB_{L(X_p, E)} + \frac{C}{s} B_{L(X_r, E)}$.

If $(x_{ij}) \in M_n(MAX(X))$ then

$$||(x_{ij})||_k = \sup \left|\left|\left(f_{uv}(x_{ij})\right)\right|\right|_{M_{nm}},$$

where the 'sup' runs over all $(f_{uv}) \in L(X, M_m)$ with $||(f_{uv})||_{L(X_k, M_m)} \leqslant 1$ and all $m \in \mathbb{N}$.

If X is Fréchet and $\dim E < +\infty$ then $L(X, E) = (X \otimes_\pi E')'$.

$\left(\overline{\Gamma(U_k \otimes B_{E'})}\right)_{k\in\mathbb{N}}$ – 0-nbhd basis in $X \otimes_\pi E'$,

$\left(\overline{\Gamma(U_k \otimes B_{E'})}\right)^\circ = B_{L(X_k,E)}$.

If $X \otimes_\pi E' \in (DN)$ then $B_{L(X_q,E)} \subset sB_{L(X_p,E)} + \frac{C}{s} B_{L(X_r,E)}$.

If $E = M_m$

If $(x_{ij}) \in M_n(MAX(X))$ then

$$||(x_{ij})||_k = \sup \left|\left|\left(f_{uv}(x_{ij})\right)\right|\right|_{M_{nm}},$$

where the 'sup' runs over all $(f_{uv}) \in L(X, M_m)$ with $||(f_{uv})||_{L(X_k,M_m)} \leqslant 1$ and all $m \in \mathbb{N}$.

If X is Fréchet and $\dim E < +\infty$ then $L(X, E) = (X \otimes_\pi E')'$.

$\left(\overline{\Gamma(U_k \otimes B_{E'})}\right)_{k\in\mathbb{N}}$ – 0-nbhd basis in $X \otimes_\pi E'$,

$\left(\overline{\Gamma(U_k \otimes B_{E'})}\right)^{\circ} = B_{L(X_k,E)}$.

If $X \otimes_\pi E' \in (DN)$ then $B_{L(X_q,E)} \subset sB_{L(X_p,E)} + \frac{C}{s} B_{L(X_r,E)}$.

If $E = M_m$ then $||(x_{ij})||_q \leqslant \sup\Big\{ \left|\left|\left((sg_{uv} + Cs^{-1}h_{uv})(x_{ij})\right)\right|\right|_{M_{nm}} :$

$(g_{uv}) \in B_{L(X_p,M_m)}, (h_{uv}) \in B_{L(X_r,M_m)} \Big\}$

If $(x_{ij}) \in M_n(MAX(X))$ then

$$||(x_{ij})||_k = \sup \left|\left|\left(f_{uv}(x_{ij})\right)\right|\right|_{M_{nm}},$$

where the 'sup' runs over all $(f_{uv}) \in L(X, M_m)$ with $||(f_{uv})||_{L(X_k,M_m)} \leqslant 1$ and all $m \in \mathbb{N}$.

If X is Fréchet and $\dim E < +\infty$ then $L(X, E) = (X \otimes_\pi E')'$.

$\left(\overline{\Gamma(U_k \otimes B_{E'})}\right)_{k\in\mathbb{N}}$ – 0-nbhd basis in $X \otimes_\pi E'$,

$\left(\overline{\Gamma(U_k \otimes B_{E'})}\right)^{\circ} = B_{L(X_k,E)}$.

If $X \otimes_\pi E' \in (DN)$ then $B_{L(X_q,E)} \subset sB_{L(X_p,E)} + \frac{C}{s} B_{L(X_r,E)}$.

If $E = M_m$ then $||(x_{ij})||_q \leqslant \sup\Big\{ \left|\left|\left((sg_{uv} + Cs^{-1}h_{uv})(x_{ij})\right)\right|\right|_{M_{nm}} :$

$(g_{uv}) \in B_{L(X_p,M_m)}, (h_{uv}) \in B_{L(X_r,M_m)} \Big\} \leqslant s||(x_{ij})||_p + Cs^{-1}||(x_{ij})||_r.$

If $(x_{ij}) \in M_n(MAX(X))$ then

$$||(x_{ij})||_k = \sup \left|\left|\left(f_{uv}(x_{ij})\right)\right|\right|_{M_{nm}},$$

where the 'sup' runs over all $(f_{uv}) \in L(X, M_m)$ with $||(f_{uv})||_{L(X_k, M_m)} \leqslant 1$ and all $m \in \mathbb{N}$.
If X is Fréchet and $\dim E < +\infty$ then $L(X, E) = (X \otimes_\pi E')'$.
$\left(\overline{\Gamma(U_k \otimes B_{E'})}\right)_{k\in\mathbb{N}}$ – 0-nbhd basis in $X \otimes_\pi E'$,
$\left(\overline{\Gamma(U_k \otimes B_{E'})}\right)^\circ = B_{L(X_k, E)}$.
If $X \otimes_\pi E' \in (DN)$ then $B_{L(X_q,E)} \subset sB_{L(X_p,E)} + \frac{C}{s} B_{L(X_r,E)}$.
If $E = M_m$ then $||(x_{ij})||_q \leqslant \sup\Big\{ \left|\left|\left((sg_{uv} + Cs^{-1}h_{uv})(x_{ij})\right)\right|\right|_{M_{nm}} :$

$$(g_{uv}) \in B_{L(X_p, M_m)}, (h_{uv}) \in B_{L(X_r, M_m)} \Big\} \leqslant s||(x_{ij})||_p + Cs^{-1}||(x_{ij})||_r.$$

Finally,

$$||(x_{ij})||_q^2 \leqslant 4C||(x_{ij})||_p||(x_{ij})||_r.$$

Proposition (P., 2011)

$X \in (DN) \;\Leftrightarrow\; MAX(X) \in (oDN).$

Proposition (P., 2011)

$X \in (DN) \ \Leftrightarrow \ MAX(X) \in (oDN)$.

Formally missing:

$X \in (DN)$ and $\dim E < +\infty \ \Rightarrow \ X \otimes_\pi E \in (DN)$.

Proposition (P., 2011)

$X \in (DN) \;\Leftrightarrow\; MAX(X) \in (oDN).$

Formally missing:
$X \in (DN)$ and $\dim E < +\infty \;\Rightarrow\; X \otimes_\pi E \in (DN).$

Corollary

$X \in (\Omega) \;\Leftrightarrow\; MIN(X) \in (o\Omega).$

Proposition (P., 2011)

Suppose H is a Fréchet-Hilbert space.

1. $H \in (DN) \;\Leftrightarrow\; H_r \in (oDN) \;\Leftrightarrow\; H_c \in (oDN)$.
2. $H \in (\Omega) \;\Leftrightarrow\; H_r \in (o\Omega) \;\Leftrightarrow\; H_c \in (o\Omega)$.

Proposition (P., 2011)

Suppose H is a Fréchet-Hilbert space.

1. $H \in (DN) \;\Leftrightarrow\; H_r \in (oDN) \;\Leftrightarrow\; H_c \in (oDN)$.
2. $H \in (\Omega) \;\Leftrightarrow\; H_r \in (o\Omega) \;\Leftrightarrow\; H_c \in (o\Omega)$.

Conjecture

1. $H \in (DN) \;\Leftrightarrow\; OH \in (oDN)$.
2. $H \in (\Omega) \;\Leftrightarrow\; OH \in (o\Omega)$.