Noncommutative analogues of Stein algebras

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Motivations from Noncommutative Geometry

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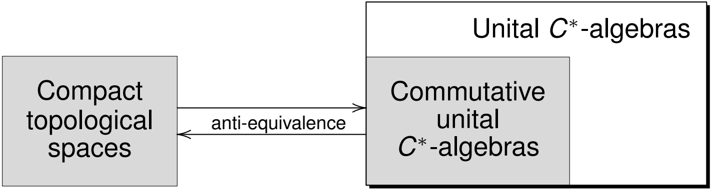
- **Philosophy:** We may think of $\mathscr{A}$ as a category of “noncommutative spaces”.

Concrete illustration: Gelfand-Naimark Theorem



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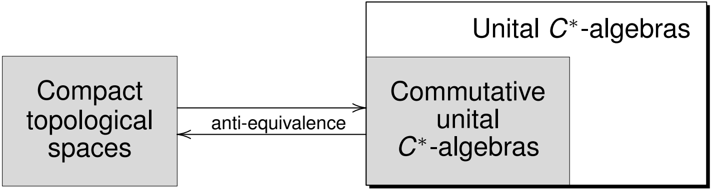

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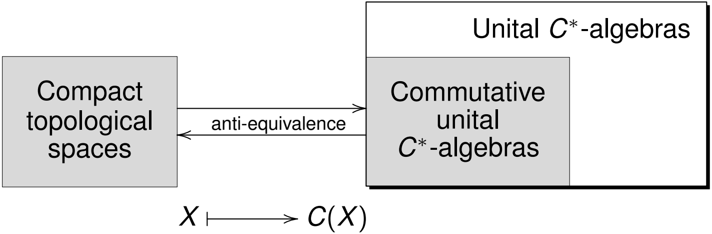
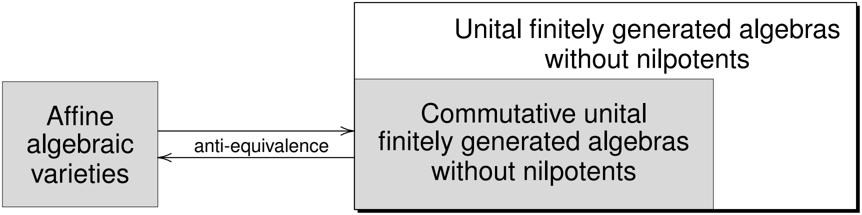


- **Philosophy:** We may think of unital C^*-algebras as "noncommutative compact topological spaces".
- This leads to Noncommutative Topology.

Concrete illustration: Nullstellensatz



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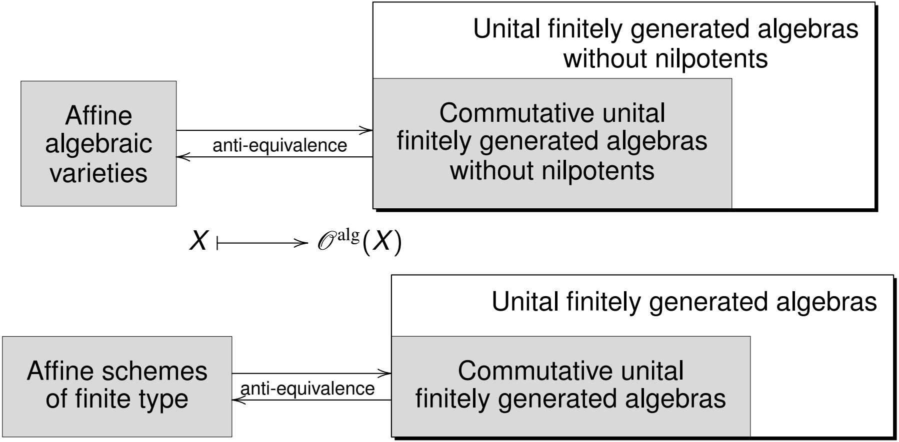

Concrete illustration: Nullstellensatz

Unital finitely generated algebras without nilpotents

Affine algebraic varieties $\underset{\text{anti-equivalence}}{\rightleftarrows}$ Commutative unital finitely generated algebras without nilpotents

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Affine schemes of finite type $\underset{\text{anti-equivalence}}{\rightleftarrows}$ Commutative unital finitely generated algebras

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- **Philosophy:** We may think of unital finitely generated algebras as "noncommutative affine algebraic varieties" (or "schemes").
- This leads to Noncommutative Affine Algebraic Geometry.

Concrete illustration: Forster's Theorem

Definition

A Fréchet algebra A is a Stein algebra if A is isomorphic to the algebra $\mathscr{O}^{\mathrm{hol}}(X)$ of holomorphic functions on a Stein space $(X, \mathscr{O}_X^{\mathrm{hol}})$.

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Theorem (O. Forster)

$$\boxed{\text{Stein spaces}} \xrightleftharpoons[\text{anti-equivalence}]{} \boxed{\text{Stein algebras}}$$

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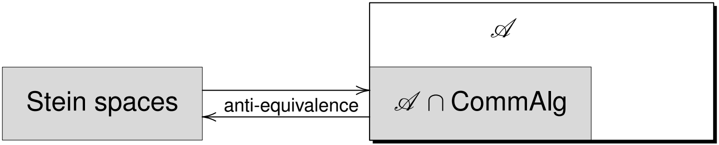
Question: what are noncommutative analogues of Stein algebras?

Goal

- **Goal:** construct a category $\mathscr{A}$ of Fréchet algebras such that



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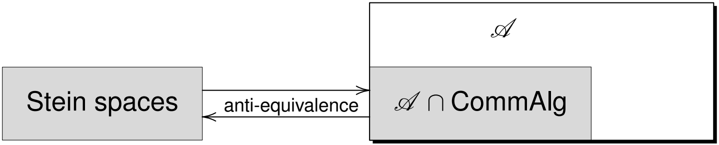

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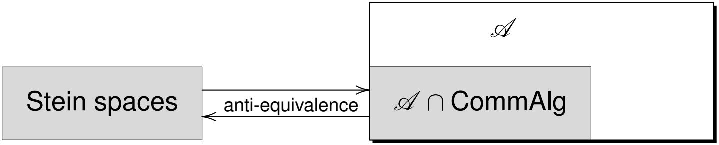

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- **Hope:** this may help to develop Noncommutative Complex Analytic Geometry.
- So far we can do it only for Stein spaces of finite embedding dimension.

Embedding dimension

Example

Let X be a complex manifold. Then X has finite embedding dimension, and $\operatorname{emb.dim}_p X = \dim X$ for all $p \in X$.

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Theorem (E. Bishop, R. Narasimhan)

Let X be a Stein space. The following conditions are equivalent:

- *X has finite embedding dimension;*
- *there exists $N \in \mathbb{N}$ and a closed analytic subspace $Y \subset \mathbb{C}^N$ such that X is biholomorphically equivalent to Y.*

Conventions

- Algebra $=$ unital $\mathbb{C}$-algebra;
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Definition

An Arens-Michael algebra is a complete topological algebra A such that the topology on A can be defined by a family $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ of submultiplicative seminorms (i.e., $\|ab\|_\lambda \le \|a\|_\lambda \|b\|_\lambda$ for all $a, b \in A$).

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Definition (J. L. Taylor, 1972)

The free analytic algebra is

$$\mathscr{F}_n = \Big\{ a = \sum_{\alpha \in W_n} c_\alpha \zeta_\alpha : \|a\|_\rho = \sum_{\alpha \in W_n} |c_\alpha| \rho^{|\alpha|} < \infty\ \forall \rho > 0 \Big\}.$$

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- $\mathscr{F}_n$ is a Fréchet Arens-Michael algebra containing F_n as a dense subalgebra.

Free holomorphic functional calculus (J. L. Taylor)

Fact

Let A be an Arens-Michael algebra. Then for each $a = (a_1, \ldots, a_n) \in A^n$ there exists a unique continuous homomorphism

$$\gamma_a^{\text{free}} \colon \mathscr{F}_n \to A \quad \text{such that } \zeta_i \mapsto a_i \ (i = 1, \ldots, n).$$

Specifically, if $f = \sum_{\alpha \in W_n} c_\alpha \zeta_\alpha \in \mathscr{F}_n$, then

$$\gamma_a^{\text{free}}(f) = f(a) = \sum_{\alpha \in W_n} c_\alpha a_\alpha,$$

where $a_\alpha = a_{\alpha_1} \cdots a_{\alpha_k}$ for $\alpha = (\alpha_1, \ldots, \alpha_k) \in W_n$.

Holomorphically closed subalgebras

Let A be a Fréchet Arens-Michael algebra.

Definition

A subalgebra $B \subset A$ is holomorphically closed if for each $b \in B^n$ and each $f \in \mathscr{F}_n$ we have $f(b) \in B$.

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Proposition

Suppose that A is commutative. Then $B \subset A$ is holomorphically closed if and only if for each $b \in B^n$ and each $f \in \mathscr{O}(\mathbb{C}^n)$ we have $f(b) \in B$.

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- If B is a Fréchet Arens-Michael algebra under a stronger topology than the topology inherited from A, then B is holomorphically closed.
- For instance, if M is a smooth compact manifold, then $C^\infty(M)$ is holomorphically closed in $C(M)$.

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- A is holomorphically generated by S if $\mathrm{Hol}(S) = A$.

HFG algebras

Proposition

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Definition

A is holomorphically finitely generated (HFG) if $A = \mathrm{Hol}(S)$ for some finite subset $S \subset A$.

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Proposition

Let A be a Fréchet Arens-Michael algebra. The following conditions are equivalent:

- *A is holomorphically finitely generated;*
- *A is isomorphic to $\mathscr{F}_n/I$ for some n and for some closed $I \triangleleft \mathscr{F}_n$.*

HFG algebras

Theorem

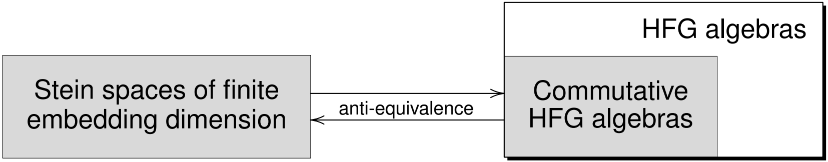
The following properties of a commutative Fréchet Arens-Michael algebra A are equivalent:

- *A is holomorphically finitely generated;*
- *A is isomorphic to $\mathscr{O}(X)$ for some Stein space X of finite embedding dimension.*

HFG algebras

Corollary

The functor $X \mapsto \mathscr{O}(X)$ is an anti-equivalence between the category of Stein spaces of finite embedding dimension and the category of commutative HFG algebras.



$$X \longmapsto \mathscr{O}(X)$$

Arens-Michael envelopes

Definition

Let A be an algebra. An Arens-Michael envelope of A is a pair $(\widehat{A}, i_A)$:

- $\widehat{A}$ is an Arens-Michael algebra;
- $i_A\colon A \to \widehat{A}$ is a homomorphism

such that

- for each Arens-Michael algebra B and
- for each homomorphism $\varphi\colon A \to B$

there exists a unique continuous homomorphism $\widehat{\varphi}\colon \widehat{A} \to B$ making the following diagram commutative:

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- J. L. Taylor (1972): "completed locally *m*-convex envelopes";
- modern terminology: A. Ya. Helemskii.

Arens-Michael envelopes

Remarks

- $\widehat{A}$ is unique.

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- The correspondence $A \mapsto \widehat{A}$ is natural (the Arens-Michael functor).

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- The correspondence $A \mapsto \widehat{A}$ is natural (the Arens-Michael functor).
- For each algebra A and each Arens-Michael algebra B we have

$$\mathrm{Hom}_{\mathrm{Alg}}(A, B) \cong \mathrm{Hom}_{\mathrm{AM}}(\widehat{A}, B).$$

Arens-Michael envelopes

Examples

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If A is a finitely generated algebra, then $\widehat{A}$ is holomorphically finitely generated.

$$\boxed{\begin{array}{c}\text{Noncommutative schemes}\\ \text{of finite type}\end{array}} \xrightarrow[\text{functor}]{\text{Arens-Michael}} \boxed{\begin{array}{c}\text{Noncommutative Stein spaces}\\ \text{of finite embedding dimension}\end{array}}$$

Example: quantum affine space

Let $q \in \mathbb{C} \setminus \{0\}$.

Definition

The (algebra of regular functions on the) quantum affine space is

$$\mathscr{O}_q^{\mathrm{alg}}(\mathbb{C}^n) = \mathbb{C}\big\langle x_1, \ldots, x_n \,\big|\, x_i x_j = q x_j x_i,\ i < j \big\rangle.$$

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Definition

The algebra of holomorphic functions on the quantum affine space is

$$\mathscr{O}_q^{\mathrm{hol}}(\mathbb{C}^n) = \mathscr{O}_q(\mathbb{C}^n) \overset{\mathrm{def}}{=} \big(\mathscr{O}_q^{\mathrm{alg}}(\mathbb{C}^n)\big)\widehat{}\,.$$

Example: quantum affine space

- Define

$$w_q\colon \mathbb{Z}_+^n \to \mathbb{R}_+, \quad w_q(\alpha) = \begin{cases} 1 & \text{if } |q| \geq 1, \\ |q|^{\sum_{i<j} \alpha_i \alpha_j} & \text{if } |q| < 1. \end{cases}$$

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$$\mathscr{O}_q^{\mathrm{hol}}(\mathbb{C}^n) = \Big\{ a = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x^\alpha : \|a\|_\rho = \sum_{\alpha \in \mathbb{Z}_+^n} |c_\alpha| w_q(\alpha) \rho^{|\alpha|} < \infty \ \forall \rho > 0 \Big\}$$

(where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.)
The topology on $\mathscr{O}_q^{\mathrm{hol}}(\mathbb{C}^n)$ is given by $\{\| \cdot \|_\rho : \rho > 0\}$.

Arens-Michael envelopes: more examples

There are similar explicit descriptions of the Arens-Michael envelopes for some other finitely generated algebras (P., 2004), including

- quantum tori;
- quantum Weyl algebras;
- quantum 2×2-matrices;
- enveloping algebras of some Lie algebras of small dimension.

The analytic free product

Definition

Let A, B be Arens-Michael algebras. The analytic free product $A \mathbin{\widehat{*}} B$ is the coproduct of A and B in the category of Arens-Michael algebras.

$$
\begin{tikzcd}
 & A \mathbin{\widehat{*}} B \arrow[dd, dashed, "\exists!"] & \\
A \arrow[ur, "j_A"] \arrow[dr, "\varphi"'] & & B \arrow[ul, "j_B"'] \arrow[dl, "\psi"] \\
 & C &
\end{tikzcd}
$$

$$\mathrm{Hom}_{\mathrm{AM}}(A \mathbin{\widehat{*}} B, C) \cong \mathrm{Hom}_{AM}(A, C) \times \mathrm{Hom}_{\mathrm{AM}}(B, C).$$

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- $A \mathbin{\widehat{*}} B$ is unique;
- $A \mathbin{\widehat{*}} B$ exists and can be constructed explicitly (J. Cuntz, 1997).

The analytic free product

Example

$\mathscr{F}_n \mathbin{\widehat{*}} \mathscr{F}_m \cong \mathscr{F}_{n+m}$. In particular,

$$\mathscr{F}_n \cong \mathscr{F}_1 \mathbin{\widehat{*}} \cdots \mathbin{\widehat{*}} \mathscr{F}_1 \cong \mathscr{O}(\mathbb{C}) \mathbin{\widehat{*}} \cdots \mathbin{\widehat{*}} \mathscr{O}(\mathbb{C}).$$

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Proposition

If A and B are HFG algebras, then so is $A \widehat{*} B$.

Holomorphic functions on the free polydisk

Let $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$.

Definition

The algebra of holomorphic functions on the free n-polydisk of radius R is

$$\mathscr{F}(\mathbb{D}_R^n) = \mathscr{O}(\mathbb{D}_R) \mathbin{\widehat{*}} \cdots \mathbin{\widehat{*}} \mathscr{O}(\mathbb{D}_R).$$

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- $\mathrm{Hom}_{\mathrm{AM}}(\mathscr{F}(\mathbb{D}_R^n), A) = \big\{a \in A^n : \sigma(a_i) \subset \mathbb{D}_R \; \forall i = 1, \ldots, n\big\}$.

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- $\mathscr{F}(\mathbb{D}_R^n)$ can be described explicitly as the set of all series $\sum_{\alpha \in W_n} c_\alpha \zeta_\alpha$ satisfying some summability conditions.

Holomorphic functions on the quantum polydisk

Recall that

$$\mathscr{O}_q^{\mathrm{hol}}(\mathbb{C}^n) = \Big\{ a = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x^\alpha : \|a\|_\rho = \sum_{\alpha \in \mathbb{Z}_+^n} |c_\alpha| w_q(\alpha) \rho^{|\alpha|} < \infty \ \forall \rho > 0 \Big\}.$$

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If $q = 1$, then $\mathscr{O}_q^{\mathrm{hol}}(\mathbb{D}_R^n) \cong \mathscr{O}^{\mathrm{hol}}(\mathbb{D}_R^n)$.

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Proposition

$$\mathscr{O}_q^{\mathrm{hol}}(\mathbb{D}_R^n) = \mathscr{F}(\mathbb{D}_R^n) / \overline{(\zeta_i \zeta_j = q \zeta_j \zeta_i, \ i < j)}.$$

As a corollary, $\mathscr{O}_q^{\mathrm{hol}}(\mathbb{D}_R^n)$ is an HFG algebra.

Polynomial functions on the quantum ball

Fix $0 < q < 1$.

Definition

$\mathrm{Pol}_q(\mathbb{C}^n)$ is the $*$-algebra generated by $x_1, \ldots, x_n$ and relations

$$\begin{aligned} x_i x_j &= q x_j x_i \quad (i < j); \\ x_i^* x_j &= q x_j x_i^* \quad (i \neq j); \\ x_i^* x_i &= q^2 x_i x_i^* + (1 - q^2)\Big(1 - \sum_{k>i} x_k x_k^*\Big). \end{aligned}$$

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- W. Pusz, S. L. Woronowicz (1989): “twisted CCR”.
- $\mathrm{Pol}_q(\mathbb{C}^n)$ is one of the basic examples in the theory of “quantum bounded symmetric domains”
(L. Vaksman, O. Bershtein, Y. Kolisnyk, D. Proskurin, S. Shklyarov, S. Sinel’shchikov, A. Stolin, L. Turowska. . . ; 1998–present time).

Continuous functions on the quantum ball

- For $k \in \mathbb{N}$, let $[k] = 1 + q^2 + q^4 + \cdots + q^{2(k-1)} = \frac{1-q^{2k}}{1-q^2}$.

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Theorem (Pusz, Woronowicz)

There exists a faithful irreducible $*$-representation $\pi \colon \operatorname{Pol}_q(\mathbb{C}^n) \to \mathscr{B}(H)$ uniquely determined by

$$\pi(x_j)e_\alpha = \sqrt{1-q^2}\sqrt{[\alpha_j+1]}\,q^{\sum_{k>j}\alpha_k}|\alpha_1, \ldots, \alpha_j + 1, \ldots, \alpha_n\rangle.$$

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The completion of $\mathrm{Pol}_q(\mathbb{C}^n)$ w.r.t. the norm $\|a\|^\infty = \|\pi(a)\|$ is denoted by $C_q(\overline{\mathbb{B}^n})$ and is called the algebra of continuous functions on the quantum ball.

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- $C_q(\overline{\mathbb{B}^n})$ may be viewed as a q-analogue of the algebra $C(\overline{\mathbb{B}^n})$ of continuous functions on the closed unit ball $\overline{\mathbb{B}^n}$ in $\mathbb{C}^n$.

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- For $n = 1$, $C_q(\overline{\mathbb{B}^n})$ was introduced by S. Klimek and A. Lesniewski (1993).

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- $\mathscr{O}_q^{\mathrm{hol}}(\mathbb{B}^n)$ is a Fréchet Arens-Michael algebra.

Holomorphic functions on the quantum ball

- The subalgebra of $\mathrm{Pol}_q(\mathbb{C}^n)$ generated by $x_1, \ldots, x_n$ is $\mathscr{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$.
- For each $r > 0$ consider $\gamma_r \in \mathrm{Aut}(\mathscr{O}_q(\mathbb{C}^n))$ given by $\gamma_r(x_i) = rx_i$ $(i = 1, \ldots, n)$.
- Define a norm on $\mathscr{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$ by $\|a\|_r^\infty = \|\gamma_r(a)\|^\infty$.
- If $q = 1$, then $\|a\|_r^\infty = \sup\limits_{z \in \overline{\mathbb{B}_r^n}} |a(z)|$, where $\overline{\mathbb{B}_r^n} = \{z \in \mathbb{C}^n : \sum |z_i|^2 \le r^2\}$.

Definition

The completion of $\mathscr{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$ w.r.t. the family $\{\|\cdot\|_r^\infty : 0 < r < 1\}$ of norms is denoted by $\mathscr{O}_q^{\mathrm{hol}}(\mathbb{B}^n)$ and is called the algebra of holomorphic functions on the quantum ball.

- $\mathscr{O}_q^{\mathrm{hol}}(\mathbb{B}^n)$ is a Fréchet Arens-Michael algebra.
- If $q = 1$, then $\mathscr{O}_q^{\mathrm{hol}}(\mathbb{B}^n) = \mathscr{O}^{\mathrm{hol}}(\mathbb{B}^n)$, the algebra of holomorphic functions on the open unit ball $\mathbb{B}^n \subset \mathbb{C}^n$.

Quantum ball vs. quantum polydisk

Recall that $\mathscr{O}_q^{\mathrm{hol}}(\mathbb{D}^n)$ is the completion of $\mathscr{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$ w.r.t. the family $\{\|\cdot\|_r^1 : 0 < r < 1\}$ of norms given by

$$\|a\|_r^1 = \sum_{\alpha\in\mathbb{Z}_+^n} |c_\alpha| w_q(\alpha) r^{|\alpha|} \quad \Big(a = \sum_{\alpha\in\mathbb{Z}_+^n} c_\alpha x^\alpha \in \mathscr{O}_q^{\mathrm{alg}}(\mathbb{C}^n)\Big).$$

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Theorem

For each $0 < q < 1$ the algebras $\mathscr{O}_q^{\mathrm{hol}}(\mathbb{B}^n)$ and $\mathscr{O}_q^{\mathrm{hol}}(\mathbb{D}^n)$ are topologically isomorphic.

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Key lemma

For each $0 < \rho < r < 1$ and each $a \in \mathscr{O}_q^{\mathrm{alg}}(\mathbb{C}^n)$ we have

$$\left(\frac{r^2-\rho^2}{r^2}\prod_{j=1}^{\infty}(1-q^{2j})\right)^{\frac{n}{2}} \|a\|_\rho^1 \le \|a\|_r^\infty \le \|a\|_r^1.$$

Therefore $\{\|\cdot\|_r^1 : 0 < r < 1\}$ and $\{\|\cdot\|_r^\infty : 0 < r < 1\}$ are equivalent.