Continuity ideals of homomorphisms from $\mathcal{C}_0(\Omega)$

Hung Le Pham

Victoria University of Wellington, New Zealand

Waterloo, August 2011

Summary

- Structure of a homomorphism θ from $\mathcal{C}_0(\Omega)$ into a Banach algebra.
- Describe the structure of the continuity ideal of θ.
- The complexity of the prime ideal structure of $\mathcal{C}_0(\Omega)$.

Preliminaries

- Let Ω be a locally compact space.
- $\mathcal{C}_0(\Omega)$ is the space of all continuous functions f vanishing at infinity on Ω, i.e.

$$\{t \in \Omega : \ |f(t)| \geq \varepsilon\} \text{ is compact for every } \varepsilon > 0.$$

- $\mathcal{C}_0(\Omega)$ is an algebra with pointwise operations.
- $\mathcal{C}_0(\Omega)$ is a Banach algebra with uniform norm

$$|f|_\Omega := \sup\{|f(t)| : \ t \in \Omega\}.$$

Kaplansky's theorem

Theorem (I. Kaplansky, 1949)

For each algebra norm $\|\cdot\|$ *on* $\mathcal{C}_0(\Omega)$, *we have*

$$\|f\| \geq |f|_\Omega \quad \textit{for every} \quad f \in \mathcal{C}_0(\Omega).$$

Consequence:

- Each *complete* algebra norm on $\mathcal{C}_0(\Omega)$ is equivalent to the uniform norm $|\cdot|_\Omega$.

Another consequence of Kaplansky's theorem

- Let θ be a *continuous* homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra B.
- Then $\theta(\mathcal{C}_0(\Omega))$ must be a closed subalgebra of B.
- $\theta(\mathcal{C}_0(\Omega))$ is isomorphic to $\mathcal{C}_0(\Omega)/\ker(\theta)$ as Banach algebras.
- $\theta(\mathcal{C}_0(\Omega))$ must has the form $\mathcal{C}_0(\Gamma)$ for some locally compact space Γ.
- Hence, each *continuous* homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra is essentially a quotient map onto its range.

Kaplansky's conjecture

Conjecture

- Every algebra norm on $\mathcal{C}_0(\Omega)$ is equivalent to the uniform norm $|\cdot|_\Omega$.
- Equivalently, every homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra is continuous.

- If $\theta : \mathcal{C}_0(\Omega) \to B$ is a discontinuous homomorphism, then $f \mapsto |f|_\Omega + \|\theta(f)\|$ is an algebra norm not equivalent to $|\cdot|_\Omega$.

Theorem (G. Dales (1979) and J. Esterle (1978))

Assuming CH, *for each infinite locally compact space* Ω, *there exists a discontinuous homomorphism from* $\mathcal{C}_0(\Omega)$ *into a Banach algebra.*

Ideals in commutative Banach algebra A

- An ideal I is modular if and only if A/I is unital.
- The **radical** of A, denoted by $\mathrm{Rad}\, A$, is

$$\mathrm{Rad}\, A = \left\{ a \in A : \ r(a) := \lim \|a^n\|^{1/n} = 0 \right\}.$$

- A is said to be **radical** if $\mathrm{Rad}\, A = A$.
- A proper ideal P is **prime** if $ab \notin P$ whenever $a, b \in A \setminus P$.
- A proper ideal I is **semiprime** if $a^2 \notin I$ whenever $a \in A \setminus I$,
  $\Leftrightarrow$ I is the intersection of a collection of prime ideals.

Ideals in $\mathcal{C}_0(\Omega)$

- $M_p = \{f \in \mathcal{C}_0(\Omega) : \ f(p) = 0\}$.
- $J_p = \{f \in \mathcal{C}_0(\Omega) : \ f(x) = 0 \text{ on a neighbourhood of } p\}$.
- $\Omega^\flat$ is **the one-point compactification** of Ω. *We always adjoin one more point ∞ to Ω.*
- $M_\infty = \mathcal{C}_0(\Omega)$ and $J_\infty = \mathcal{C}_c(\Omega)$.
- For a prime ideal P, $\exists$ a unique $p \in \Omega^\flat$ such that $J_p \subseteq P \subseteq M_p$.
- P is modular if and only if $p \in \Omega$.

An important fact on prime ideals in $\mathcal{C}_0(\Omega)$

- If P and Q_1, Q_2 are prime ideals such that $P \subseteq Q_i$, then either $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$.
- Equivalently: in $\mathcal{C}_0(\Omega)$, the collection of prime ideals that contain a given prime ideal is a chain!
- A consequence: if a semiprime ideal I contains the intersection of n prime ideals, then I itself is the intersection of n prime ideals.

The continuity ideals

Let A and B be commutative Banach algebras.
Let $\theta : A \to B$ be a homomorphism.
Then the **continuity ideal of θ** is defined as

$$\mathcal{I}(\theta) = \{a \in A : \; x \mapsto \theta(ax) \text{ is continuous}\}$$

- $\mathcal{I}(\theta)$ is an ideal, measuring the continuity of θ.
- $I \subseteq \mathcal{I}(\theta)$ for each ideal I in A such that θ is continuous on I.
- If $A = \mathcal{C}_0(\Omega)$ then θ is continuous on $\mathcal{I}(\theta)$.

Structure of (discontinuous) homomorphisms

Theorem (W. Bade and P. Curtis, 1960; with some improvement by J. Esterle and A. Sinclair)

Let $\theta : \mathcal{C}_0(\Omega) \to B$ be a (discontinuous) homomorphism. Then

1. *$\mathcal{I}(\theta)$ is the largest ideal of $\mathcal{C}_0(\Omega)$ on which θ is continuous.*
2. *There exists a (non-empty) finite subset $\{p_1, \ldots, p_n\}$ of $\Omega^\flat$ such that*
$$\bigcap_{i=1}^{n} J_{p_i} \subseteq \mathcal{I}(\theta) \subseteq \bigcap_{i=1}^{n} M_{p_i}.$$
3. *There exists a continuous homomorphism $\mu : \mathcal{C}_0(\Omega) \to B$ such that $\mu = \theta$ on a dense subalgebra of $\mathcal{C}_0(\Omega)$ containing $\mathcal{I}(\theta)$.*
4. *Set $\nu = \theta - \mu$. Then ν maps into $\mathrm{Rad}\, B$, and the restriction of ν to $\bigcap_{i=1}^{n} M_{p_i}$ is a homomorphism ν' onto a dense subalgebra of $\mathrm{Rad}\, B$ such that $\mathcal{I}(\theta) = \ker \nu'$.*

Theorem (cont.)

5. *There exist linear maps* $\nu_1, \ldots, \nu_n : \mathcal{C}_0(\Omega) \to \mathrm{Rad}\, B$ *such that*
   1. $\nu = \nu_1 + \cdots + \nu_n$,
   2. *each* $\nu_i | M_{p_i}$ $(1 \leq i \leq n)$ *is a non-zero homomorphism, and*
   3. $\nu_i(\mathcal{C}_0(\Omega)) \cdot \nu_j(\mathcal{C}_0(\Omega)) = \{0\}$ *for each* $1 \leq i \neq j \leq n$.
6. *The ideals* $\ker \theta$ *and* $\mathcal{I}(\theta)$ *are always intersections of prime ideals.*
7. *In the case where B is radical, then* $\mathcal{I}(\theta) = \ker \theta$ *is always an intersection of non-modular prime ideals.*

The existence of discontinuous homomorphisms revisited

Theorem (Dales and Esterle)

Assuming CH.

1. *Let P be any nonmodular prime ideal in $\mathcal{C}_0(\Omega)$ such that $|\mathcal{C}_0(\Omega)/P| = \mathfrak{c}$. Then there exists a homomorphism θ from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra R such that $\ker\theta = P$.*
2. *Let P be any prime ideal in $\mathcal{C}_0(\Omega)$ such that $|\mathcal{C}_0(\Omega)/P| = \mathfrak{c}$. Then there exists a homomorphism θ from $\mathcal{C}_0(\Omega)$ into a Banach algebra B such that $\mathcal{I}(\theta) = P$.*

The role of CH:

- (R. Solovay and H. Woodin) There is a model of ZFC where every homomorphism from $\mathcal{C}_0(\Omega)$ is continuous.
- (H. Woodin, 1993) There is a model of ZFC $+ \neg$CH such that, for every infinite locally compact space Ω, there exists a discontinuous homomorphism from $\mathcal{C}_0(\Omega)$.

Continuity ideals and finite intersections of primes

Corollary

Assuming CH. *Let I be any ideal in* $\mathcal{C}_0(\Omega)$ *such that I is a finite intersection of prime ideals and such that* $|\mathcal{C}_0(\Omega)/I| = \mathfrak{c}$. *Then there exists a homomorphism* θ *from* $\mathcal{C}_0(\Omega)$ *into a Banach algebra B such that* $\mathcal{I}(\theta) = I$.

- Is $\mathcal{I}(\theta)$ always a finite intersection of prime ideals?
- (J. Esterle, 1978) $\mathcal{I}(\theta)$ is always a finite intersection of prime ideals when Ω is an F-space. E.g. when Ω is extremely disconnected.

Is $\mathcal{I}(\theta)$ always a finite intersection of prime ideals?

Theorem (P, 2008)

Let Ω be a metrizable locally compact space.

1. *Suppose that $\Omega^\flat$ has "finite limit level". Then $\mathcal{I}(\theta)$ is a finite intersection of prime ideals for every homomorphism θ from $\mathcal{C}_0(\Omega)$ into another Banach algebra.*
2. *Suppose that $\Omega^\flat$ has "infinite limit level". Assuming CH. Then there exists a homomorphism θ from $\mathcal{C}_0(\Omega)$ into a Banach algebra B such that $\mathcal{I}(\theta)$ is not a finite intersection of prime ideals.*

- If $\Omega^\flat$ is metrizable and has "finite limit level" then Ω is countable.
- $\mathbb{R}^\flat$ has "infinite limit level".

Pseudo-finite families of prime ideals in $\mathcal{C}_0(\Omega)$

Let Ω be a locally compact space.

Definition

A family $(P_i)_{i\in I}$ of prime ideals in $\mathcal{C}_0(\Omega)$ is **pseudo-finite** if

$$f \in \bigcup_{i\in I} P_i \quad \Rightarrow \quad f \in P_i \text{ for all but finitely many } i \in I.$$

- If I is infinite then $\bigcup_{i\in I} P_i$ is a prime ideal in $\mathcal{C}_0(\Omega)$.
- If J is an infinite subset of I then $\bigcup_{i\in J} P_i = \bigcup_{i\in I} P_i$.

Compact families of prime ideals in $\mathcal{C}_0(\Omega)$

Definition

- A family $\mathfrak{C}$ of prime ideals in $\mathcal{C}_0(\Omega)$ is **relatively compact** if every sequence in $\mathfrak{C}$ contains a pseudo-finite subsequence.
- A family $\mathfrak{C}$ of prime ideals in $\mathcal{C}_0(\Omega)$ is **compact** if every sequence in $\mathfrak{C}$ contains a pseudo-finite subsequence whose union also belongs to $\mathfrak{C}$.

There is a topology on the collection of prime ideals on $\mathcal{C}_0(\Omega)$ such that:

- Each pseudo-finite sequence is a convergent sequence.
- Each (relatively) compact family is a (relatively) sequentially compact set.

Continuity ideals and compact families of prime ideals

For each ideal I in $\mathcal{C}_0(\Omega)$ and each $f \in \mathcal{C}_0(\Omega)$, set

$$I{:}f := \{g \in \mathcal{C}_0(\Omega) : \; fg \in I\}$$

Theorem (P, 2010)

Let $\theta : \mathcal{C}_0(\Omega) \to B$ be a homomorphism. Set $\mathfrak{P}$ be the collection of prime ideals of the form $\mathcal{I}(\theta){:}f$ for some $f \in \mathcal{C}_0(\Omega)$. Then

- $\mathcal{I}(\theta) = \bigcap \{P : \; P \in \mathfrak{P}\}$;
- $\mathfrak{P}$ *is a relatively compact family of prime ideals in* $\mathcal{C}_0(\Omega)$.

Remarks:

- When B is radical, $\mathcal{I}(\theta) = \ker(\theta)$.
- Moreover, when B is radical, prime ideals of the form $\ker(\theta){:}f$ must be nonmodular.
- In general, $\ker(\theta) = \mathcal{I}(\theta) \cap \text{a closed ideal}$.

The closure of a relatively compact family

Let $\mathfrak{P}$ be any relatively compact family of prime ideals in $\mathcal{C}_0(\Omega)$.
Define $\mathfrak{Q}$ to be the family of all ideals in $\mathcal{C}_0(\Omega)$ that are unions of countably many ideals in $\mathfrak{P}$.
Obviously:

- Every ideal in $\mathfrak{Q}$ is a prime ideal.
- $\mathfrak{Q}$ contains $\mathfrak{P}$ as well as the unions of all pseudo-finite sequences in $\mathfrak{P}$.
- $\bigcap\{P\colon\ P \in \mathfrak{P}\} = \bigcap\{P\colon\ P \in \mathfrak{Q}\}$.

It turns out that:

- $\mathfrak{Q}$ is a compact family of prime ideals.
- $\mathfrak{Q}$ is the collection of the unions of all pseudo-finite sequences in $\mathfrak{P}$.
- $\Rightarrow$ $\mathfrak{Q}$ is the smallest compact family of prime ideals that contains $\mathfrak{P}$.

The structure of a compact family of prime ideals

Let $\mathfrak{Q}$ be any compact family of prime ideals in $\mathcal{C}_0(\Omega)$.

- Every chain in $\mathfrak{Q}$ is well-ordered.
- In particular, every nonempty chain in $\mathfrak{Q}$ has the smallest element.
- Take $\mathfrak{P}_0$ be the collection of minimal elements of $\mathfrak{Q}$. Then $\mathfrak{P}_0$ is relatively compact and

$$\bigcap \{P\colon\ P \in \mathfrak{P}_0\} = \bigcap \{P\colon\ P \in \mathfrak{Q}\}\,.$$

- $\mathfrak{P}_0$ is “non-redundant” in the sense that, for each $Q \in \mathfrak{P}_0$,

$$\bigcap \{P\colon\ P \in \mathfrak{P}_0 \setminus \{Q\}\} \not\subseteq Q.$$

  In fact, $\bigcap \{P\colon\ P \in \mathfrak{Q} \setminus \{Q\}\} \not\subseteq Q$.
- $\{\text{Prime ideals of the form } \mathcal{I}(\theta)\!:\!f\}$ is “non-redundant”.

The structure of a compact family of prime ideals (cont.)

- Since prime ideals in $\mathcal{C}_0(\Omega)$ that contain a given prime ideal form a chain, $\mathfrak{Q}$ in fact has the form of a “downward” forest.
- Lets call the union of a maximal chain in $\mathfrak{Q}$ one of its **roof**.
- A roof must either be a prime ideal in $\mathcal{C}_0(\Omega)$ or $\mathcal{C}_0(\Omega)$ itself.
- $\mathfrak{Q}$ always has only finitely many roofs (i.e. the forest has only finitely many “downward” trees).

The (partial) converse

Theorem (P, 2010)

Let I be the intersection of a relatively compact family $\mathfrak{P}$ of prime ideals in $C_0(\Omega)$ with the properties that every chain in the closure of $\mathfrak{P}$ is countable *and that*

$$|C_0(\Omega)/I| = \mathfrak{c}.$$

Assuming CH, then there exists a homomorphism θ from $C_0(\Omega)$ into a Banach algebra such that $\mathcal{I}(\theta) = I$.

- The closure $\mathfrak{Q}$ of $\mathfrak{P}$ is the smallest compact family of prime ideals that contains $\mathfrak{P}$.
- Every chain in $\mathfrak{Q}$ is well-ordered.
- If $\mathfrak{P}$ is countable, then every chain in $\mathfrak{Q}$ is countable.
- If I is the intersection of a countable family of prime ideals, then $\mathfrak{P}_0$ is countable, and so I is the intersection of a countable, relatively compact family of prime ideals.

Proof:

- Since the closure $\mathfrak{Q}$ has only finitely many roofs, we need only consider the case when $\mathfrak{Q}$ has a single roof.
- Using this, we may further reduce to the case when all prime ideals in $\mathfrak{Q}$ is nonmodular.
- Thus we only need to prove the following.

Theorem (P, 2010)

Let I be the intersection of a relatively compact family $\mathfrak{P}$ of nonmodular prime ideals in $\mathcal{C}_0(\Omega)$ with the properties that every chain in the closure of $\mathfrak{P}$ is countable *and that*

$$|\mathcal{C}_0(\Omega)/I| = \mathfrak{c}.$$

Assuming CH, then there exists a homomorphism θ from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra such that $\ker(\theta) = I$.

Proof: the main construction

- Let $I, \mathfrak{P}$ be as in the previous slide.

Proposition

Then there exist a prime ideal $\mathcal{P}$ in $c_0 = \mathcal{C}_0(\mathbb{N})$ and, for each $P \in \mathfrak{P}$, a homomorphism $\theta_P : \mathcal{C}_0(\Omega) \to c_0/\mathcal{P}$ such that

- $\ker \theta_P = P$ for every $P \in \mathfrak{P}$;
- the set $\{\theta_P(f) \mid P \in \mathfrak{P}\}$ is finite for each $f \in \mathcal{C}_0(\Omega)$.

- Based on the construction of Dales and Woodin for the case when $|\mathfrak{P}| = 1$ so that I is a prime ideal itself.
- Assuming CH, by Dales and Esterle's theorem, there exists an embedding $\iota : c_0/\mathcal{P} \hookrightarrow R$ into a radical Banach algebra.
- $f \mapsto (\iota \circ \theta_P(f))_{P \in \mathfrak{P}}$ is then a homomorphism from $\mathcal{C}_0(\Omega)$ into (the radical of) $\ell^\infty(\mathfrak{P}, R)$ with kernel precisely I.

Proof: the role of compactness in the precedding construction

- In the construction of $\{\theta_P : \; P \in \mathfrak{P}\}$ instead of the condition that
  - the set $\{\theta_P(f) \mid \; P \in \mathfrak{P}\}$ is finite for each $f \in \mathcal{C}_0(\Omega)$

  we actually only need to aim for the weaker condition that
  - the set $\{\theta_{P_n}(f) \mid \; P \in \mathfrak{P}\}$ is finite for each $f \in \mathcal{C}_0(\Omega)$ and each pseudo-finite sequence (P_n) in $\mathfrak{P}$.
- If, for some $f \in \mathcal{C}_0(\Omega)$, the set $\{\theta_P(f) \mid \; P \in \mathfrak{P}\}$ were infinite, then, by the relative compactness of $\mathfrak{P}$, there existed a pseudo-finite sequence (P_n) in $\mathfrak{P}$ such that $\theta_{P_n}(f)$ $(n \in \mathbb{N})$ are all distinct.

The role of countability I

- The hypothesis that every chain in $\mathfrak{Q}$ is countable is required because of the limitation in our proof.
- For simplicity, say we have a well-ordered chain indexed by ordinals smaller than a given ordinal.
- We want to construct inductively, for each ordinal α, a subalgebra A_α of $\mathcal{C}_0(\Omega)$ such that:
  - $A_\alpha \supseteq A_\beta$ when $\alpha \leq \beta$;
  - A_α is “sufficiently big”.
- If we have A_α we can construct $A_{\alpha+1}$.
- However, if we have $A_\alpha, A_{\alpha+1}, \ldots, A_{\alpha+n}$ $(n \in \mathbb{N})$, ..., how to construct $A_{\alpha+\omega}$? Is $\bigcap_{n\in\mathbb{N}} A_{\alpha+n}$ necessarily nonempty?
- If we construct $A_{\alpha+\omega}$ first, then there is no problem. $\Rightarrow$ We can only deal with countable ordinals.

Problem

Can we eliminate this limitation?

The role of countability II

- If the ideal $I = \bigcap \mathfrak{P}$ is the intersection of a countable family of prime ideals, then it is the intersection of a countable, relatively compact family of prime ideals.
- In this case, every chain in the closure of this countable, relatively compact family of prime ideals is countable.
- Question: Is the continuity ideal of a homomorphism $\theta : \mathcal{C}_0(\Omega) \to B$ always a countable intersection of prime ideals?
- Answer: No.

Theorem (P, 2009)

Suppose that Ω is metrizable and has "infinite limit level" (e.g. $\Omega = \mathbb{R}$), then there exists a "nonredundant" pseudo-finite family $(P_i)_{i\in\mathfrak{c}}$ of prime ideals in $\mathcal{C}_0(\Omega)$; i.e.

- *If $f \in \bigcup_{i\in\mathfrak{c}} P_i$, then $f \in P_i$ for all but finitely many $i \in \mathfrak{c}$;*
- *$\bigcap_{i\in\mathfrak{c},\ i\neq\alpha} P_i \nsubseteq P_\alpha$, for each $\alpha \in \mathfrak{c}$;*

- Note that any pseudo-finite family of prime ideals is relatively compact and every chain in its closure has length at most 2.
- Thus a previous theorem can be strengthened as follows.

Theorem

Let Ω be a metrizable locally compact space.

1. *Suppose that $\Omega^\flat$ has "finite limit level". Then $\mathcal{I}(\theta)$ is a finite intersection of prime ideals for every homomorphism θ from $\mathcal{C}_0(\Omega)$ into another Banach algebra.*
2. *Suppose that $\Omega^\flat$ has "infinite limit level". Assuming CH. Then there exists a homomorphism θ from $\mathcal{C}_0(\Omega)$ into a Banach algebra B such that $\mathcal{I}(\theta)$ is not a countable intersection of prime ideals.*

The role of countability III

- Let $\mathfrak{P}$ be a relatively compact family of prime ideals in $\mathcal{C}_0(\Omega)$.
- We may suppose that $\mathfrak{P}$ is "non-redundant".
- Set $\mathfrak{Q}$ be the closure of $\mathfrak{P}$.

Question

Is every chain in $\mathfrak{Q}$ necessarily countable?

- Recall that every chain in $\mathfrak{Q}$ is necessarily well-ordered.
- "Cheat" question: Is a well-ordered chain of prime ideals in $\mathcal{C}_0(\Omega)$ necessarily countable?
- (M. Mandelker, 1968) There exits a well-ordered chain of prime ideals in $\mathcal{C}_0(\mathbb{R})$ order isomorphic to κ for every countable ordinal κ.

- Answer to the "cheat" question: No.

Theorem (P, 2009)

Suppose that Ω is metrizable and has "infinite limit level", then there exists a well-ordered chain of prime ideals in $\mathcal{C}_0(\Omega)$ that is order-isomorphic to $\mathfrak{c}$.

- In fact, if Ω is metrizable, separable, and uncountable (e.g. $\mathbb{R}$), then, for every ordinal κ of cardinality $\mathfrak{c}$, there exists a well-ordered chain of prime ideals in $\mathcal{C}_0(\Omega)$ that is order-isomorphic to κ.

Question

- Is every chain in $\mathfrak{Q}$ necessarily countable?
- The same question but only for those $\mathfrak{P}$ that (known to) arises from the continuity ideals.

Examples of compact families of prime ideals

- Finite unions of pseudo-finite families of prime ideals are relatively compact. More complicated example:
- Let κ be an ordinal, considered with the order topology.
- Suppose that κ has "finite limit level" $\Rightarrow$ κ is countable.
- Enlarging κ if necessary, we may suppose further that κ is compact.
- Suppose that Ω be a metrisable and has "infinite limit level".
- Then there exists a family of prime ideals $(P_\alpha : \alpha \in \kappa)$ in $\mathcal{C}_0(\Omega)$ satisfying the following:
  1. $(P_{\alpha_n} : n \geq n_0)$ is a pseudo-finite sequence with union P_α for some $n_0 \in \mathbb{N}$ whenever (α_n) converges to α in the order topology of κ;
  2. a condition guarantees the non-redundancy.

- It follows that $(P_\alpha : \alpha \in \kappa)$ is a compact family of prime ideals such that
$$\bigcap_{\alpha\in\kappa} P_\alpha$$
is never an intersection of a finite union of pseudo-finite families of prime ideals.
- Varying κ, we can make $(P_\alpha : \alpha \in \kappa)$ to contain chain of any *finite* length.

Let $\mathfrak{P}$ be a “non-redundant” relatively compact family of prime ideals in $\mathcal{C}_0(\Omega)$ and set $\mathfrak{Q}$ be the closure of $\mathfrak{P}$.

Question

- Is every chain in $\mathfrak{Q}$ necessarily finite?
- The same question but only for those $\mathfrak{P}$ that (known to) arises from the continuity ideals.

Prime z-filters and prime z-ideals

- Let $f \in \mathcal{C}(\Omega)$. Then $\mathbf{Z}(f) := \{t \in \Omega \mid f(t) = 0\}$.
- $\mathbf{Z}[\Omega] := \{\mathbf{Z}(f) \mid f \in \mathcal{C}(\Omega)\}$.
- A linear subspace I of $\mathcal{C}_0(\Omega$ is a z**-ideal** if

$$f \in I,\ g \in \mathcal{C}_0(\Omega),\ \mathbf{Z}(g) \supseteq \mathbf{Z}(f) \quad \Rightarrow \quad g \in I.$$

- A z**-filter** $\mathcal{F}$ on Ω is a nonempty proper subset of $\mathbf{Z}[\Omega]$ s.t.

$$\begin{aligned} Z_1 \text{ and } Z_2 \in \mathcal{F} \quad &\Rightarrow \quad Z_1 \cap Z_2 \in \mathcal{F}, \\ Z_1 \in \mathcal{F},\ Z_2 \in \mathbf{Z}[\Omega],\ Z_2 \supseteq Z_1 \quad &\Rightarrow \quad Z_2 \in \mathcal{F}. \end{aligned}$$

- A z-filter $\mathcal{F}$ is a **prime z-filter** if

$$Z_1 \text{ and } Z_2 \in \mathbf{Z}[\Omega] \setminus \mathcal{F} \quad \Rightarrow \quad Z_1 \cup Z_2 \notin \mathcal{F}.$$

- Set of (prime) z-filters $\leftrightarrow$ Set of (prime) z-ideals

$$\mathcal{F} \quad \leftrightarrow \quad \{f \in \mathcal{C}_0(\Omega) \mid \mathbf{Z}(f) \in \mathcal{F}\}.$$

A reduction to prime z-filters

- Let $\mathfrak{P}$ be a "non-redundant" relatively compact family of prime ideals in $\mathcal{C}_0(\Omega)$.
- Set $\mathfrak{Q}$ be the closure of $\mathfrak{P}$.
- Then:
  1. Every ideal in $\mathfrak{Q} \setminus \mathfrak{P}$ is a prime z-ideal.
  2. Each $P \in \mathfrak{P}$ contains a prime z-ideal P_z such that

     $$\{P_z \mid P \in \mathfrak{P}\}$$

     is a "non-redundant" relatively compact family of prime z-ideals with closure $\mathfrak{Q}$.
- Thus previous questions can reduce to corresponding questions on prime z-filters.

Further question

- (Bade and Curtis) Homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras are built from homomorphisms from $\mathcal{C}_0(\Omega)$ into radical Banach algebras.
- Suppose that $\theta : \mathcal{C}_0(\Omega) \to R$ is a radical homomorphism. Then $\ker(\theta)$ is the intersection of a relatively compact family of nonmodular prime ideals.
- Conversely, given an ideal I which is the intersection of a relatively compact family $\mathfrak{P}$ of nonmodular prime ideals in $\mathcal{C}_0(\Omega)$ (with some additional conditions – at least for now). We can construct a radical homomorphism of a very specific form:

- First, there exist a radical Banach algebra R_0, and, for each $P \in \mathfrak{P}$, a homomorphism $\varphi_P : \mathcal{C}_0(\Omega) \to R_0$ such that
  - $\ker \varphi_P = P$ for every $P \in \mathfrak{P}$;
  - the set $\{\varphi_P(f) \mid P \in \mathfrak{P}\}$ is finite for each $f \in \mathcal{C}_0(\Omega)$.
- Then, the map $f \mapsto (\varphi_P(f))_{P \in \mathfrak{P}}$ is a homomorphism from $\mathcal{C}_0(\Omega)$ into $\mathrm{Rad}\, \ell^\infty(\mathfrak{P}, R_0)$ with kernel precisely I.

Question

Let $\theta : \mathcal{C}_0(\Omega) \to R$ be a radical homomorphism and let $\mathfrak{P}$ be the relatively compact family of prime ideals associated with $\ker(\theta) = \mathcal{I}(\theta)$. Does there exists another radical Banach algebra R_0 such that

1. $\mathrm{Rad}\, \ell^\infty(\mathfrak{P}, R_0) \subseteq R$; and
2. θ is formed from a family $(\varphi_P)_{P \in \mathfrak{P}}$ of homomorphisms into R_0 as above.

General commutative algebras

Let A and B be commutative Banach algebras.
Let $\theta : A \to B$ be a homomorphism.
In general, $\mathcal{I}(\theta)$ is not necessarily an intersection of prime ideals.
The **prime radical** of $\mathcal{I}(\theta)$ is

$$\sqrt{\mathcal{I}(\theta)} := \{a \in A : \ a^n \in \mathcal{I}(\theta) \text{ for some } n \in \mathbb{N}\}\,.$$

$\sqrt{\mathcal{I}(\theta)}$ is the intersection of all prime ideals that contain $\mathcal{I}(\theta)$.

Theorem

Let $\mathfrak{P}$ be the collection of minimal ideals among the prime ideals of the form $\mathcal{I}(\theta){:}a$ for some $a \in A$. Then

- $\sqrt{\mathcal{I}(\theta)} = \bigcap\{P : \ P \in \mathfrak{P}\}$;
- $\mathfrak{P}$ *is a relatively compact family of prime ideals in A.*

Theorem

Let $\theta : A \to B$ *be an epimorphism. Then*

- $\sqrt{\mathcal{I}(\theta)}$ *is a* finite *intersection of prime ideals of the form* $\mathcal{I}(\theta)$:*a*.
- *There exists* $k \in \mathbb{N}$ *such that*

$$\sqrt{\mathcal{I}(\theta)} = \left\{ a \in A \colon \; a^k \in \mathcal{I}(\theta) \right\}.$$

Abstract continuity ideals

Let $\theta : A \to B$ be a homomorphism between two commutative Banach algebras.
Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in A. Then

$$\mathcal{I}(\theta){:}a_1 a_2 \cdots a_n \subseteq \mathcal{I}(\theta){:}a_1 a_2 \cdots a_{n+1} \qquad (\forall n \in \mathbb{N}).$$

This is true for any ideal not just for $\mathcal{I}(\theta)$.
However, by the well-known stability lemma, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{I}(\theta){:}a_1 a_2 \cdots a_n = \mathcal{I}(\theta){:}a_1 a_2 \cdots a_{n+1} \qquad (\forall n \geq n_0).$$

Definition

An ideal I of a commutative algebra A is an **abstract continuity ideal** if, for each sequence (a_n) in A, there exists $n_0 \in \mathbb{N}$ such that

$$I{:}a_1 a_2 \cdots a_n = I{:}a_1 a_2 \cdots a_{n+1} \qquad (\forall n \geq n_0).$$

Theorem

Let A be a commutative algebra and I an ideal of A. Then:

1. *If I is an abstract continuity ideal, then $\sqrt{I}$ is the intersection of a relatively compact family of prime ideals of the form $I{:}a$.*
2. *If I is the intersection of a relatively compact family of prime ideals in A, then I is an abstract continuity ideal.*

Corollary

Let Ω be a locally compact space and I an ideal of $\mathcal{C}_0(\Omega)$. Then TFAE:

1. *I is an abstract continuity ideal;*
2. *I is the intersection of a relatively compact family of prime ideals in $\mathcal{C}_0(\Omega)$;*