Operator algebras from commuting semigroup actions

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This is joint work with Benton Duncan.

Setting

Let $\mathcal{S}$ be an abelian semigroup, with cancellation, containing 0, and X a compact metric space.

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The 'polynomial algebra' $\mathcal{A}_0$ consists of (finite) formal sums

$$F = \sum_{t \in \mathcal{S}} S_t \, f_t$$

where $f_t \in C(X)$, and the elements S_t and f satisfy the commutation relation

$$f \, S_t = S_t \, f \circ \sigma_t.$$

Fix $x \in X$ we define a 'left regular representation' π_x on the algebra $\mathcal{A}_0$ on $\ell_2(\mathcal{S})$.
Let $\xi_s \in \ell_2(\mathcal{S})$ be the function

$$\xi_s(t) = \begin{cases} 1, & \text{if } t = s \\ 0, & \text{otherwise.} \end{cases}$$

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Define π_x by $\pi_x(f)\xi_t = f(\sigma_t(x))\xi_t,\ f \in C(X)$ and $\pi_x(S_s)\xi_t = \xi_{s+t}$.
Then π_x is an isometric covariant representation of $\mathcal{A}_0$ and the family of representations $\pi_x,\ x \in X$ separates the points of $\mathcal{A}_0$.

Tensor algebra

We can define a norm on $\mathcal{A}_0$ by

$$||F|| = \sup_{x \in X} ||\pi_x(F)||$$

The completion of $\mathcal{A}_0$ in this norm, which we denote by $\mathcal{A}$ or $\mathcal{A}(\mathcal{S}, X)$, is called the *tensor algebra*

Proposition

There is a faithful, completely contractive conditional expectation $P_0 : \mathcal{A} \to C(X)$.

Orbit cocycles

Orbit representations are defined in a manner similar to left regular representations, but the underlying Hilbert space is $\ell_2(\text{orbit})$ rather that $\ell_2(\mathcal{S})$.

In order to define these representations, we must first introduce *orbit cocycles*.

Fix $x \in X$ and let $\mathcal{S}(x) = \{\sigma_t(x) : \ t \in \mathcal{S}\}$ be the orbit of the point x under the action of the semigroup. A map $\mu : \mathcal{S} \times \mathcal{S}(x) \to \mathbb{C}$ is an orbit cocycle if it satisfies

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- For each $t \in \mathcal{S}$ and any $y \in \mathcal{S}(x)$

$$\sum_{\sigma_t y_j = \sigma_t(y)} |\mu(t, y_j)|^2 \leq 1$$

- (cocycle condition)

$$\mu(s + t, y) = \mu(t, y)\mu(s, \sigma_t(y))$$

Orbit representations

Fix $x \in X$ and let $y \in \mathcal{S}(x)$. Define the function $\xi_y(w) = 1$ if $w = y$ and 0 otherwise. Now fix an orbit cocycle μ and define the orbit representation ρ_μ on this basis by

$$\begin{aligned} \rho_\mu(f)\xi_y &= f(y)\xi_y \quad \text{for } f \in C(X), \text{ and} \\ \rho_\mu(S_t)\xi_y &= \mu(t, y)\xi_{\sigma_t(y)} \end{aligned}$$

A calculation shows that ρ_μ is a contractive covariant representation.

invariant subspace

To see the relationship between the two classes of representations, fix $x \in X$. We assume that for y in the orbit of x, $\{t \in \mathcal{S} : \ \sigma_t(x) = y\}$ is finite.
Consider the map

$$\xi_t \to \xi_{\sigma_t(x)}$$

and extend to linear combinations. If $\mathcal{H}_0$ is the closed subspace of $\ell_2(\mathcal{S})$ which is mapped to 0, we can show $\mathcal{H}_0$ is invariant under the representation π_x.

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Thus if Q is the orthogonal projection of $\ell_2(\mathcal{S})$ onto $\mathcal{H}_0$, the space $\mathcal{H}_1 := Q^{\perp}\ell_2(\mathcal{S})$ is semi-invariant. Thus, one can define a representation

$$\pi_x^1(F) = Q^{\perp}\pi_x(F)|\mathcal{H}_1, \ F \in \mathcal{A}.$$

left-regular orbit cocycle

We now give an example of an orbit cocycle, called the *left-regular orbit cocycle*. Define

$$\mu(t, y) = \frac{||\pi_x^1(S_t)Q^\perp \xi_u||}{||Q^\perp \xi_u||} = \frac{||Q^\perp \xi_{t+u}||}{||Q^\perp \xi_u||}$$

if $y = \sigma_u(x)$.
One shows this is well defined, and satisfies the two conditions for an orbit cocycle.

left-regular orbit representation

There is a unitary $W : \ell_2(\mathcal{S}(x)) \to \mathcal{H}_1$ such that

$$W^* \pi_x^1(F) W = \rho_{x,\mu}(F), \;\; F \in \mathcal{A}$$

where μ is the left-regular orbit cocycle.

definition

Recall that if $\mathcal{A}$ is an operator algebra, the C*-envelope of $\mathcal{A}$, $\mathrm{C}^*(\mathcal{A})$ is a C*-algebra characterized as follows: there is a completely isometric embedding

$$j : \mathcal{A} \to \mathrm{C}^*(\mathcal{A})$$

whose image generated $\mathrm{C}^*(\mathcal{A})$ (as a C*-algebra). Furthermore, if $\mathcal{C}$ is a C*-algebra and $\omega : \mathcal{A} \to \mathcal{C}$ is a completely isometric embedding, then there is a surjective map $\zeta : \mathcal{C} \to \mathrm{C}^*(\mathcal{A})$ such that $j = \zeta \circ \omega$.

C*-envelope of the tensor algebra

There is a description of the C*-envelope of the tensor algebra using C*-correspondences, due to Katsura and Muhly & Solel. This was adapted by Davidson & Katsoulis in their memoir on free actions on operator algebras.

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There is a description of the C*-envelope of the tensor algebra using C*-correspondences, due to Katsura and Muhly & Solel. This was adapted by Davidson & Katsoulis in their memoir on free actions on operator algebras.
We will need a 'working definition' of the C*-envelope in order to show our mail results: that the left-regular representations are Shilov, and the left-regular orbit representations have a Shilov resolution. We present another approach to the C*-envelope, which is possible in our context.

Extensions of dynamical systems

The second construction of the C*-envelope requires us to consider "extensions" of dynamical systems. An extension of the system $(X, \mathcal{S}, \sigma)$ is another dynamical system $(Y, \mathcal{S}, \tau)$ together with a continuous surjection $p : Y \to X$ such that the diagram

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$$\begin{array}{ccc} Y & \xrightarrow{\tau_t} & Y \\ p\big\downarrow & & p\big\downarrow \\ X & \xrightarrow{\sigma_t} & X \end{array}$$

commutes, for all $t \in \mathcal{S}$.

homeomorphism extensions

Since the (abelian) semigroup $\mathcal{S}$ is a semigroup with cancellation, it is easy to obtain the enveloping group, namely the smallest abelian group $\mathcal{G}$ which contains $\mathcal{S}$. That can be expressed as $\mathcal{G} = \mathcal{S} - \mathcal{S}$. However, the group $\mathcal{S}$ need not have any connection with the dynamical system $(X, \mathcal{S}, \sigma)$ since the maps σ_t may not be invertible.

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Proposition

There is an extension $(\widetilde{X}, \mathcal{S}, \widetilde{\sigma})$ of the dynamical system $(X, \mathcal{S}, \sigma)$ for which the maps τ_t are homeomorphisms, $t \in \mathcal{S}$. Furthermore, this extension is "minimal".

main results

From the above, we can consider the group $\mathcal{G}$ as a dynamical system acting on $\widetilde{X}$.

Theorem

The C^*-envelope of the tensor algebra can be identified with the crossed product $C(\widetilde{X}) \times_{\widetilde{\sigma}} \mathcal{G}$.

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From the above, we can consider the group $\mathcal{G}$ as a dynamical system acting on $\widetilde{X}$.

Theorem

The C^*-envelope of the tensor algebra can be identified with the crossed product $C(\widetilde{X}) \times_{\widetilde{\sigma}} \mathcal{G}$.

Furthermore, we obtain that the representations π_x of the tensor algebra are Shilov representations. Specifically, there is a representation $\widetilde{\pi}_x$ of the crossed product on the Hilbert space $\ell_2(\mathcal{G})$ such that

$$\widetilde{\pi}_x(F)|\ell_2(\mathcal{S}) = \pi_x(F), \quad \text{for } F \in \mathcal{A}.$$

shilov resolution

While the left regular orbit representations π_x^1 need not be Shilov, they have a Shilov resolution. This follows from the fact that the π_x are Shilov, and we have the resolution of Hilbert modules

$$(0) \to \mathcal{H}_0 \to \ell_2(\mathcal{S}) \to \mathcal{H}_1 \to (0).$$