Compactness and weak-star continuity of derivations on weighted convolution algebras

Banach Algebras 2011, Waterloo, Canada

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Overview

Motivation

- Traditionally the study of derivations from a Banach algebra to its Banach modules has mainly focused on the existence of such derivations.
- In some recent papers by Choi and Heath:
  - Characterise the derivations from a concrete Banach algebra to its dual space.
  - Use this characterisation to study properties of the derivations.
  - [Choi and Heath, 2010]: Weakly compact derivations from $l^1(\mathbb{Z}^+)$ to its dual space.
  - [Choi and Heath, 2011]: Compact derivations from the disc algebra to its dual space.
- We continue this line of thinking and consider properties of derivations from weighted convolution algebras $L^1(\omega)$ on $\mathbb{R}^+$ to their dual spaces.

Definitions

- Let ω be a continuous weight function on $\mathbb{R}^+$, that is, a positive and continuous function on $\mathbb{R}^+$ satisfying $\omega(0) = 1$ and $\omega(t+s) \leq \omega(t)\omega(s)$ for all $t, s \in \mathbb{R}^+$.

- With the usual convolution product the weighted space

$$L^1(\omega) = \{f : \mathbb{R}^+ \to \mathbb{C} \;:\; \|f\| = \int_0^\infty |f(t)|\omega(t)\,dt < \infty\}$$

  is a commutative Banach algebra.

- The space $M(\omega)$ of locally finite, complex Borel measures μ on $\mathbb{R}^+$ with

$$\|\mu\| = \int_0^\infty \omega(t)\,d|\mu|(t) < \infty$$

  is also a Banach algebra under convolution. Moreover,

$$M(\omega) = \mathrm{Mul}(L^1(\omega)).$$

Definitions – continued

- Let

$$L^\infty(1/\omega) = \{\varphi : \mathbb{R}^+ \to \mathbb{C} \ : \ \|\varphi\| = \operatorname{ess\,sup}_{t \in \mathbb{R}^+} |\varphi(t)|/\omega(t) < \infty\}.$$

Then

$$L^\infty(1/\omega) = L^1(\omega)^*.$$

- Subspaces of $L^\infty(1/\omega)$:

$$C_0(1/\omega) \leq C_b(1/\omega) \leq L^\infty(1/\omega).$$

We have

$$M(\omega) = C_0(1/\omega)^*.$$

- The dual space $L^\infty(1/\omega) = L^1(\omega)^*$ is a Banach $L^1(\omega)$-module:

$$\langle f, g \cdot \varphi \rangle = \langle f * g, \varphi \rangle \qquad \text{for } f, g \in L^1(\omega) \text{ and } \varphi \in L^\infty(1/\omega).$$

Derivations from $L^1(\omega)$ to $L^\infty(1/\omega)$

A linear map $D : L^1(\omega) \to L^\infty(1/\omega)$ is called a **derivation** if

$$D(f*g) = f\cdot Dg + g\cdot Df \qquad \text{for } f,g \in L^1(\omega).$$

Theorem [Grønbæk, 1989] and [Bade and Dales, 1989]

- Let $\varphi \in L^\infty(1/\omega)$. Then

$$(D_\varphi f)(t) = \int_0^\infty f(s)\,\frac{s}{t+s}\,\varphi(t+s)\,ds \qquad \text{for } t\in\mathbb{R}^+ \text{ and } f\in L^1(\omega)$$

  defines a continuous derivation $D_\varphi : L^1(\omega) \to L^\infty(1/\omega)$.

- D_φ has a unique extension to a continuous derivation $\overline{D}_\varphi : M(\omega) \to L^\infty(1/\omega)$ with

$$(\overline{D}_\varphi \delta_s)(t) = \frac{s}{t+s}\,\varphi(t+s) \qquad \text{for } t,s\in\mathbb{R}^+.$$

- Conversely, every continuous derivation from $L^1(\omega)$ to $L^\infty(1/\omega)$ equals D_φ for some $\varphi \in L^\infty(1/\omega)$.

Oversigt

Weak-star continuity

Motivation

- [Grabiner, 1988 and 2010]: Every non-zero continuous homomorphism $\Phi : L^1(\omega_1) \to L^1(\omega_2)$ has a unique extension to a continuous homomorphism $\widetilde{\Phi} : M(\omega_1) \to M(\omega_2)$, and this extension is automatically weak-star continuous.
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Question For which $\varphi \in L^\infty(1/\omega)$ is

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Idea If $\overline{D}_\varphi$ is a adjoint operator, then it is weak-star continuous.

A characterisation

Definition

For $f \in L^1(\omega)$, $\varphi \in L^\infty(1/\omega)$ and $t \in \mathbb{R}^+$ we let

$$(T_\varphi f)(t) = \int_0^\infty f(s)\,\frac{t}{t+s}\,\varphi(t+s)\,ds = \langle f, \overline{D}_\varphi \delta_t \rangle.$$

Observation For $f, g \in L^1(\omega)$ and $\varphi \in L^\infty(1/\omega)$ we have

$$\langle f, D_\varphi g \rangle = \langle T_\varphi f, g \rangle$$

if $T_\varphi f \in C_0(1/\omega)$. (In general $\operatorname{ran} T_\varphi \subseteq C_b(1/\omega)$.)

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Proposition

For $\varphi \in L^\infty(1/\omega)$ the following conditions are equivalent:

(a) $\overline{D}_\varphi$ is weak-star continuous.

(b) $\overline{D}_\varphi \delta_t / \omega(t) \to 0$ weak-star in $L^\infty(1/\omega)$ as $t \to \infty$.

(c) $\operatorname{ran} T_\varphi \subseteq C_0(1/\omega)$.

(d) $\operatorname{ran} T_\varphi \subseteq C_0(1/\omega)$ and $T_\varphi^* = \overline{D}_\varphi$.

A characterisation – continued

Heuristically we can think of $\overline{D}_\varphi \mu$ as

$$(\overline{D}_\varphi \mu)(t) = \int_0^\infty \frac{s}{t+s} \varphi(t+s)\, d\mu(s) = \int_0^\infty (\overline{D}_\varphi \delta_s)(t)\, d\mu(s),$$

although the integrals need not be defined.

Corollary

Under the equivalent conditions in the proposition we have

$$\overline{D}_\varphi \mu = \int_0^\infty \overline{D}_\varphi \delta_s \, d\mu(s)$$

as a weak-star Bochner integral in $L^\infty(1/\omega)$ for $\mu \in M(\omega)$, that is,

$$\langle f, \overline{D}_\varphi \mu \rangle = \int_0^\infty \langle f, \overline{D}_\varphi \delta_s \rangle \, d\mu(s)$$

for $f \in L^1(\omega)$ and $\mu \in M(\omega)$.

Sufficient conditions

Aim Find conditions on φ which ensure weak-star continuity of $\overline{D}_\varphi$.

Definition

For $\varphi \in L^\infty(1/\omega)$ and $t, \varepsilon \in \mathbb{R}^+$ we let

$$U_{t,\varepsilon} = \{s \in [t, t+1] : |\varphi(s)|/\omega(s) \geq \varepsilon\}.$$

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Theorem

Let $\varphi \in L^\infty(1/\omega)$ and assume that

$$m(U_{t,\varepsilon}) \to 0 \qquad \text{as } t \to \infty \qquad \text{for every } \varepsilon > 0.$$

Then $\overline{D}_\varphi$ is weak-star continuous and $\overline{D}_\varphi = T_\varphi^*$.

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Idea in the proof Since φ/ω is not bounded away from zero on large sets, the definition of $T_\varphi f$ can be used to show that $T_\varphi f \in C_0(1/\omega)$ for $f \in L^1(\omega)$.

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Question Is the condition in the theorem also necessary?

Sufficient conditions – continued

Corollary

Let $\varphi \in L^\infty(1/\omega)$ and assume that $\varphi(t)/\omega(t) \to 0$ as $t \to \infty$.

Then $\overline{D}_\varphi$ is weak-star continuous and $\overline{D}_\varphi = T_\varphi^*$.

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Then $\overline{D}_\varphi$ is weak-star continuous and $\overline{D}_\varphi = T_\varphi^*$.

Corollary

- Let (α_n) be a sequence with $0 < \alpha_n < 1$ $(n \in \mathbb{N})$ and $\alpha_n \to 0$ as $n \to \infty$.
- Let $\varphi = \sum_{n=1}^\infty 1_{[n,n+\alpha_n]} \cdot \omega \in L^\infty(1/\omega)$.

Then $\overline{D}_\varphi$ is weak-star continuous
(but we do not have $\varphi(t)/\omega(t) \to 0$ as $t \to \infty$).

Counterexamples

Proposition

- Suppose $\exists C > 0$ such that $\int_x^{x+1} \omega(y)\,dy \geq C\omega(x)$ for all $x \in \mathbb{R}^+$.
- Let (a_n) be a sequence in $\mathbb{R}^+$ with $a_0 \geq 1$ and $a_{n+1} \geq a_n + 1$ for $n \in \mathbb{N}$.
- Let $\varphi = \sum_{n=1}^{\infty} 1_{[a_n, a_n+1]} \cdot \omega \in L^\infty(1/\omega)$.

Then $\overline{D}_\varphi$ is not weak-star continuous.

In particular $\overline{D}_\omega$ is not weak-star continuous.

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Then $\overline{D}_\varphi$ is not weak-star continuous.

In particular $\overline{D}_\omega$ is not weak-star continuous.

Idea in the proof

$$\frac{\delta_{a_n}}{\omega(a_n)} \to 0 \qquad \text{weak-star in } M(\omega)$$

whereas

$$\overline{D}_\varphi\left(\frac{\delta_{a_n}}{\omega(a_n)}\right) \qquad \text{does not tend to 0 weak-star in } L^\infty(1/\omega)$$

as $n \to \infty$.

Oversigt

Compactness

Motivation [Choi and Heath, 2010 and 2011].

Questions

- For which $\varphi \in L^{\infty}(1/\omega)$ are

$$D_{\varphi} : L^{1}(\omega) \to L^{\infty}(1/\omega)$$

and

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  compact?

- Do we have

$$D_\varphi \text{ compact} \qquad \Rightarrow \qquad \overline{D}_\varphi \text{ compact?}$$

Necessary conditions

Proposition

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Let $\varphi \in L^{\infty}(1/\omega)$ be real-valued. Assume that there exists $t_0 > 0$ such that

$$\lim_{t \to (t_0)_-} \varphi(t) \qquad \text{and} \qquad \lim_{t \to (t_0)_+} \varphi(t)$$

exist and are different. Then D_{φ} and $\overline{D}_{\varphi}$ are not compact.

(A slightly stronger result can actually be proved.)

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Remark Because of these results, we will focus on $\varphi \in C_b(1/\omega)$ with $\varphi(0) = 0$.

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Ideas in the proof

- $$(\overline{D}_\varphi \mu)(t) = \langle \psi_t, \mu \rangle \qquad \text{for } \mu \in M(\omega) \text{ and } t \in \mathbb{R}^+,$$
  where $\psi_t \in C_0(1/\omega)$ is given by
  $$\psi_t(s) = \frac{s}{t+s}\varphi(t+s) \qquad \text{for } t, s \in \mathbb{R}^+.$$
- $\{\psi_t/\omega(t) : t \in \mathbb{R}^+\}$ is totally bounded in $C_0(1/\omega)$.

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Question Is it necessary to have $\varphi \in C_0(1/\omega)$?

Answer

- No, for the compactness of D_φ (see below).
- We do not know for $\overline{D}_\varphi$.

Sufficient conditions – continued

Proposition

Let $\varphi \in C_b(1/\omega)$. Assume that $\varphi(0) = 0$ and that $\overline{D}_\varphi \delta_s/\omega(s)$ has a limit in $L^\infty(1/\omega)$ as $s \to \infty$. Then D_φ is compact.

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Ideas in the proof

- $$D_\varphi f = \int_0^\infty f(s)\omega(s) \frac{\overline{D}_\varphi \delta_s}{\omega(s)} \, ds$$
  exists as a Bochner integral.
- $\{\overline{D}_\varphi \delta_s / \omega(s) : s \in \mathbb{R}^+\}$ is pre-compact in $C_0(1/\omega)$.
- Use a standard result about Bochner integrals.

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- Use a standard result about Bochner integrals.

Question Is $\overline{D}_\varphi$ compact under the assumptions in the proposition?

Sufficient conditions – continued

Proposition

Assume that $\omega(s) \geq 1$ for every $s \in \mathbb{R}^+$, $\omega(s) \to \infty$ as $s \to \infty$ and

$$\sup_{t \in \mathbb{R}^+} \frac{|\omega(t+s) - \omega(s)|}{\omega(t)\omega(s)} \to 0 \qquad \text{as } s \to \infty.$$

Let $\varphi = \omega - 1$.

Then D_φ is compact, whereas $\varphi \notin C_0(1/\omega)$.

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Idea in the proof $\overline{D}_\varphi \delta_s / \omega(s) \to 1$ in $L^\infty(1/\omega)$ as $s \to \infty$.

Corollary

Let $\alpha > 0,\ \omega(t) = (1+t)^\alpha\ (t \in \mathbb{R}^+)$ and let $\varphi = \omega - 1$.

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Sufficient conditions – continued

The condition $\varphi(t)/\omega(t) \to 0$ as $t \to \infty$ in the theorem cannot in general be relaxed to $\varphi(t)/\omega(t) \to \alpha$ as $t \to \infty$ for some $\alpha \in \mathbb{C}$:

Proposition

Let $\varphi \in L^\infty(\mathbb{R}^+)$ and assume that $\varphi(t) \to \alpha$ as $t \to \infty$ for some $\alpha \neq 0$. Then $D_\varphi : L^1(\mathbb{R}^+) \to L^\infty(\mathbb{R}^+)$ and $\overline{D}_\varphi : M(\mathbb{R}^+) \to L^\infty(\mathbb{R}^+)$ are not compact.

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Idea in the proof $\overline{D}_\varphi \delta_n \to \alpha$ weak-star in $L^\infty(\mathbb{R}^+)$, but $\|\overline{D}_\varphi \delta_n - \alpha\| \to |\alpha|$ as $n \to \infty$.