# On the Multiplier algebra of some Topological Algebras

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Banach Algebras 2011

Waterloo, August 2011

# Basic Definitions

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1. The **right (left) annihilator** of an algebra $A$ is defined as $\mathcal{A}_r(A) = \{x \in A : Ax = 0\}$ $(\mathcal{A}_l(A) = \{x \in A : xA = 0\})$. Note that these sets are in fact two-sided ideals of $A$.

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6. If $\mathfrak{S}_r(A)$, $\mathfrak{S}_l(A)$ exist and $\mathfrak{S}_r(A) = \mathfrak{S}_l(A)$, then the resulting two-sided ideal is called the **socle** of $A$ and it is denoted by $\mathfrak{S}(A)$.

### Definition

Let $A$ be an algebra.

7. A **left (right) multiplier** on $A$ is a linear mapping $T : A \longrightarrow A$ such that $T(xy) = T(x)y$ $(= xT(y))$ for all $x, y \in A$.

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8. $T$ is called a **two-sided multiplier** (or simply, a **multiplier)** on $A$ if it is a left and a right multiplier.

   We will denote by $\mathcal{M}_l(A)$ ($\mathcal{M}_r(A)$) the set of all left (right) multipliers on $A$ and by $\mathcal{M}(A)$ the set of all multipliers on $A$. Note that $\mathcal{M}(A) = \mathcal{M}_l(A) \cap \mathcal{M}_r(A)$.

   We will denote by $\mathcal{L}(A)$ the algebra of all linear operators on $A$.

Remark. $\mathcal{M}(A)$ is a subalgebra of $\mathcal{L}(A)$. The same holds for $\mathcal{M}_l(A)$ and $\mathcal{M}_r(A)$.

Let $x \in A$; the function $l_x$ given by

$$l_x(y) = xy \ , \ x \in A$$

is an operator on $A$.

Due to the associativity of multiplication on $A$, $l_x$ is a left multiplier. We can define $r_x$, the right multiplier associated to $x$, in a similar way.

# Our Results

**Theorem**

*Let $A$ be a preannihilator algebra. Then:*

1. *The mapping*

$$\begin{aligned} L : A &\rightarrow \mathcal{M}_l(A) \\ x &\longmapsto l_x \end{aligned}$$

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2. *$A$ is a left ideal of the algebra $\mathcal{M}_l(A)$.*
3. *If $B$ is a subalgebra of $\mathcal{M}_l(A)$ such that $A \subseteq B$, then $I \cap A \neq (0)$ for each non-zero right ideal $I$ of $B$.*

# Our Results

**Proof.**

(1) Since the left multiplication satisfies the relations $l_{(x+\lambda y)} = l_x + \lambda l_y$ and $l_{xy} = l_x \circ l_y$ for every $x, y \in A$ and $\lambda \in \mathbb{C}$, we get that $L$ is a homomorphism.

If $L(x) = 0$, then $l_x(y) = 0$ for all $y \in A$. Namely, $xy = 0$ for all $y \in A$. By hypothesis, $x = 0$ and $L$ is finally a monomorphism.

(2) Under the identification $x \equiv l_x$, we only have to show that $A$ absorbs multiplication on the left: If $x \in A$ and $T \in \mathcal{M}_l(A)$, then, for each $y \in A$, we have $Tl_x(y) = T(xy) = T(x)y = l_{T(x)}$ and therefore $Tl_x = l_{T(x)}$. □

# Our Results

**Proof.**

[Proof (Cont.)] Let us suppose that $A \cap I = 0$. On one hand, since $A$ is a left ideal in $\mathcal{M}_l(A)$, we have $IA \subseteq BA \subseteq \mathcal{M}_l(A)A \subseteq A$.

On the other hand, since $I$ is a right ideal in $B$, we have $IA \subseteq IB \subseteq I$. Therefore $IA \subseteq A \cap I = 0$.

Now, if $T \in I$ and $x, y \in A$, we have
$0 = (T \circ l_x)(y) = T(l_x(y)) = T(xy) = T(x)y$, that is, $T(x)A = 0$, which implies that $T(x) = 0$. Then $I = 0$. □

# Our Results

## Corollary

*Every preannihilator algebra $A$ is an essential two-sided ideal in its multiplier algebra $\mathcal{M}(A)$.*

## Corollary

*Let $A$ be a preannihilator algebra. If $B$ is a subalgebra of $\mathcal{M}_l(A)$ which contains $A$, then $\mathfrak{S}_l(A)$ is a left ideal of $B$ and $\mathfrak{S}_l(A) \subseteq \mathfrak{S}_l(B)$.*

# Our Results

**Proof.**

(Second Corollary): Clearly $A$ is a left ideal in $B$.

It is known that $Soc_l(A) = Soc_l(B) \cap A$; therefore $Soc_l(A) \subseteq Soc_l(B)$ and $B\ (Soc_l(A)) = B\ (Soc(B) \cap A) \subseteq B\ Soc(B) \cap B\ A \subseteq Soc(B) \cap A = Soc(A)$ so that $Soc_l(A)$ is a left ideal in $B$. □

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2. A **locally convex algebra** is a topological algebra $A$ which is a locally convex space. In this case its topology can be given by a family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms such that for each $\alpha \in \Lambda$ there exists $\beta \in \Lambda$ such that

$$p_\alpha(xy) \leq p_\beta(x)p_\beta(y) \tag{1}$$

for all $x$, $y \in A$.

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   (1) $$p_\alpha(xy) \leq p_\beta(x)p_\beta(y)$$
   for all $x$, $y \in A$.
3. A locally convex algebra is said to be **multiplicatively convex** (shortly **m-convex**) if every seminorm is submultiplicative i.e. (1) can be replaced by
   (2) $$p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$$
   for each $\alpha \in \Lambda$ and all $x$, $y \in A$.

Let $(A, (p_\alpha)_{\alpha\in\Lambda})$ be a complete locally $m$-convex algebra and let

$$\begin{aligned} \rho_\alpha : A &\rightarrow A/\ker p_\alpha \doteq A_\alpha \\ x &\longmapsto \rho_\alpha(x) = x + \ker p_\alpha \doteq x_\alpha, \qquad \alpha \in \Lambda \end{aligned}$$

be the respective quotient maps.

Then $\|x_\alpha\|_\alpha \doteq p_\alpha(x)$, $x \in A$ and $\alpha \in \Lambda$, defines a norm on $A_\alpha$, so that $A_\alpha$ is a normed algebra and the morphisms $\rho_\alpha$, $\alpha \in \Lambda$ are continuous.

Let $\widetilde{A_\alpha}$, $\alpha \in \Lambda$, denote the completion of $A_\alpha$ with respect to $\| \|_\alpha$.

$\Lambda$ is endowed with a partial ordering by putting $\alpha \preceq \beta$ if $p_\alpha(x) \leq p_\beta(x)$ for all $x \in A$.

Thus, $\ker p_\beta \subseteq \ker p_\alpha$ and hence the continuous onto morphism

$$\begin{aligned} f_{\alpha\beta} : A_\beta &\longrightarrow A_\alpha \\ x_\beta &\longmapsto f_{\alpha\beta}(x_\beta) \doteq x_\alpha \qquad (\alpha \preceq \beta) \end{aligned}$$

is defined.

Moreover, $f_{\alpha\beta}$ extends to a continuous morphism

$$\widetilde{f}_{\alpha\beta} : \widetilde{A_\beta} \longrightarrow \widetilde{A_\alpha}$$

Thus, $(A_\alpha, f_{\alpha\beta})$ (respectively $(\widetilde{A_\alpha}, \widetilde{f}_{\alpha\beta})$), $\alpha, \beta \in \Lambda$ with $\alpha \preceq \beta$, are projective systems of normed algebras (respectively, Banach algebras) so that:

$$A \cong \varprojlim A_\alpha \cong \varprojlim \widetilde{A_\alpha} \qquad \text{(Arens-Michael decomposition)}$$

within topological algebra isomorphisms.

Note that the projective limit algebra $\varprojlim \widetilde{A_\alpha}$ can be realized as a subalgebra of the direct product $\prod_{\alpha\in\Lambda} \widetilde{A_\alpha}$ and therefore the canonical maps $\widetilde{\pi_\beta} : \prod_{\alpha\in\Lambda} \widetilde{A_\alpha} \longrightarrow \widetilde{A_\beta}$ (for $\beta \in \Lambda$) can be restricted to $\varprojlim \widetilde{A_\alpha}$.

$$\begin{array}{ccccc} & & A \cong \varprojlim \widetilde{A_\alpha} & & \\ & \widetilde{\pi_\beta}| \swarrow & & \searrow \widetilde{\pi_\alpha}| & \\ \widetilde{A_\beta} & & \overset{\widetilde{f}_{\alpha,\beta}}{- - - - - - - - - \longrightarrow} & & \widetilde{A_\alpha} \end{array}$$

# Definitions in Topological Algebras.

## Definition

A projective system $\{(A_\alpha, f_{\alpha\beta})_{\alpha,\beta\in\Lambda}\}$ of topological algebras is called **perfect**, if the restrictions to the projective limit algebra

$$A = \varprojlim A_\alpha = \{(x_\alpha) \in \prod_{\alpha\in\Lambda} A_\alpha : f_{\alpha\beta}(x_\beta) = x_\alpha,\ \alpha \preceq \beta \text{ in } \Lambda\}$$

of the canonical projections $\pi_\alpha : \prod_{\alpha\in\Lambda} A_\alpha \longrightarrow A_\alpha$, $\alpha \in \Lambda$, namely, the (continuous algebra) morphisms

$$f_\alpha = \pi_\alpha \mid_{A=\varprojlim A_\alpha} : A \longrightarrow A_\alpha,\ \alpha \in \Lambda,$$

are onto maps.

The resulting projective limit algebra $A = \varprojlim A_\alpha$ is called a **perfect** (topological) algebra [Haralampidou, 2003].

## Definition

An **approximate identity** on a topological algebra $A$ is a net $\{e_\delta\}_{\delta\in\Delta}$ in $A$ such that

$$xe_\delta \xrightarrow[\delta]{} x \text{ and } e_\delta x \xrightarrow[\delta]{} x$$

for each $x \in A$.

## Theorem

*Let $(A, (p_\alpha)_{\alpha\in\Lambda})$ be a complete locally m-convex algebra with an approximate identity $\{e_\delta\}_{\delta\in\Delta}$. Suppose that each factor $A_\alpha = A/\ker p_\alpha$ in the Arens-Michael decomposition of $A$ is complete. Then each (two-sided) multiplier $T$ of $A$ is continuous, i.e., $\mathcal{M}(A)$ is a subalgebra of $\mathcal{L}(A)$.*

## Proof.

Let $T$ be an element in $\mathcal{M}(A)$, and $\alpha \in \Lambda$. Take $x \in \ker p_a$. For $\varepsilon > 0$, there exists an index $\delta_0 \in \Delta$ such that $p_\alpha(T(x) - T(x)e_\delta) < \varepsilon$ whenever $\delta \geq \delta_0$.

We have: $p_\alpha(T(x)) = p_\alpha(T(x - xe_{\delta_0} + xe_{\delta_0}))$
$= p_\alpha(T(x) - T(xe_{\delta_0}) + T(xe_{\delta_0})) \leq$
$\leq p_\alpha(T(x) - T(xe_{\delta_0})) + p_\alpha(T(xe_{\delta_0}))$
$= p_\alpha(T(x) - T(x)e_{\delta_0}) + p_\alpha(xT(e_{\delta_0})) \leq$
$\leq p_\alpha(T(x) - T(x)e_{\delta_0}) + p_a(x)p_\alpha(T(e_{\delta_0})) < \varepsilon.$
Since this is true for an arbitrary $\varepsilon > 0$, we conclude that $p_\alpha(T(x)) = 0$, that is, $T(x) \in \ker p_\alpha$.

Then the initial multiplier $T : A \longrightarrow A$ has projections $T_\alpha : A_\alpha \longrightarrow A_\alpha$, where $T_\alpha(x + \ker p_\alpha) = T(x) + \ker p_\alpha$, multipliers of the proper normed algebras $A_\alpha$, which by hypothesis, are Banach algebras for every $\alpha \in \Lambda$. □

## Proof (cont.)

By definition we have $T_\alpha \circ \rho_\alpha = \rho_\alpha \circ T$, where $\rho_\alpha : A \to A_\alpha$, $\alpha \in \Lambda$, are the canonical quotient maps. Moreover, $f_{\alpha\beta} \circ T_\beta = T_\alpha \circ f_{\alpha\beta}$ for all $\alpha \leq \beta$ in $\Lambda$. Here $f_{\alpha\beta} \quad (\alpha \leq \beta)$ denote the connecting maps of the projective system. Namely, $(T_\alpha)_{\alpha\in\Lambda}$ is a projective system of maps with respect to $\{(A_\alpha, f_{\alpha\beta}), \alpha \leq \beta\}$ in $\Lambda$, so that $T = \varprojlim T_\alpha$.

Denote by $f_\alpha$ the restrictions of $\pi_\alpha : \prod_{\alpha\in\Lambda} A_\alpha \to A_\alpha$ to the projective limit $\varprojlim A_\alpha$. Since $f_\alpha \circ \varphi = \rho_\alpha$, where $\varphi$ is the topological algebra isomorphism identifying $A$ with $\varprojlim A_\alpha$, we set $f_\alpha = \rho_\alpha$.

Since multipliers on proper Banach algebras are bounded (equivalently continuous), $T_\alpha$ is continuous on $A_\alpha$. Therefore $T_\alpha \circ f_\alpha$ is continuous, as well. Since $T_\alpha \circ f_\alpha = f_\alpha \circ T$ for all $\alpha \in \Lambda$, $T$ is continuous. □

# Basic Definitions in Topological Algebras

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1. A seminorm $p$ on the $^*$-algebra satisfies the $C^*$-**condition** (or it is a $C^*$-**seminorm**) if $p(x^*x) = p(x)^2$ for each $x \in A$.

   It is known that such a seminorm must be submultiplicative and $^*$-preserving.

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   It is known that such a seminorm must be submultiplicative and $^*$-preserving.

2. **A locally** $C^*$-**algebra** is a complete, locally $m$-convex, $^*$-algebra $(A, (p_\alpha)_{\alpha\in\Lambda})$ such that each $p_\alpha$ $(\alpha \in \Lambda)$ is a $C^*$-seminorm.

**Corollary**

*Let $A$ be a locally $C^*$-algebra. Then the algebra of multipliers of $A$ is a subalgebra of the algebra of continuous linear operators on $A$.*

# Some Results.

Suppose $(A, \{q_\alpha : \alpha \in \Lambda\})$ is a locally $C^*$-algebra. Then [Joița, 2005] $\mathcal{M}_l(A)$ becomes a locally convex algebra with respect to the family of seminorms $\{\widetilde{q}_\alpha : \alpha \in \Lambda\}$ defined as follows: for $T_l \in \mathcal{M}_l(A)$,

$$\widetilde{q}_\alpha(T_l) = \sup\{q_\alpha(T_l(x)),\, x \in A \text{ and } q_\alpha(x) \leq 1\}.$$

**Theorem (Joița, 2005)**

*Let $A$ be a locally $C^*$-algebra. Then we have:*

1. *$\mathcal{M}_l(A)$ is a complete locally m-convex algebra.*
2. *If $A = \varprojlim A_\alpha$ and the canonical maps $g_\alpha : A \to A_\alpha$ are all surjective, then the locally m-convex algebras $\mathcal{M}_l(A)$ and $\varprojlim \mathcal{M}_l(A_\alpha)$ are isomorphic.*

# Our Results.

**Theorem**

*Let $(A, (p_\alpha)_{\alpha\in\Lambda})$ be a complete locally m-convex $^*$-algebra with continuous involution and such that the respective projective system is perfect. Moreover, suppose that $A$ can be turned into a locally $C^*$-algebra under a weaker locally convex topology than the initial one, denoted by $(A, (q_\alpha)_{\alpha\in\Lambda})$. Then:*

$$\mathcal{M}_l((A, (q_\alpha)_{\alpha\in\Lambda})) \cong \varprojlim \mathcal{M}_l(F_\alpha),$$

*within an isomorphism, where $F_\alpha$, $\alpha \in \Lambda$, are the normed factors in the Arens-Michael decompostion of $A$ under the new topology.*

## Proof.

By hypothesis, $(q_\alpha)_{\alpha\in\Lambda}$ is the family of seminorms defining the weaker locally convex topology on $A$ (equivalently, $\ker(p_\alpha) \subseteq \ker(q_\alpha), \alpha \in \Lambda$).

This topology is actually a locally $m$-convex one. Put $F_\alpha \equiv A/\ker(q_\alpha)$. The factor normed algebras $F_\alpha$, $\alpha \in \Lambda$, in the Arens-Michael decomposition (under the new topology) are $C^*$-algebras and hence $\widetilde{F}_\alpha = F_\alpha$. Take the respective analysis

$$A \cong \varprojlim A_\alpha$$

with respect to $(p_\alpha)_{\alpha\in\Lambda}$. □

## Proof (Cont.)

By hypothesis, each canonical projection map $f_\alpha : \varprojlim A_\alpha \to A_\alpha$ is onto. Denote by

$$g_\alpha : \varprojlim F_\alpha \to F_\alpha$$

the respective projection maps corresponding to the family $(q_\alpha)$.

Since $\ker(p_\alpha) \subseteq \ker(q_\alpha)$ there is an induced surjective map $\phi_\alpha : A/\ker(p_\alpha) \to A/\ker(q_\alpha)$ given by $\phi_\alpha(x + \ker(p_\alpha)) = x + \ker(q_\alpha)$. It is easily checked that $\phi_\alpha \circ f_\alpha = g_\alpha$ (here we use the identification $f_\alpha = \rho_\alpha$). Thus the $g_\alpha$ are onto as well.

The assertion follows from Joiţa's result. □

# Definitions in Topological Algebras

**Definition (Haralampidou, 1993)**

A **locally** $m$**-convex** $H^*$**-algebra** is an algebra $A$ equiped with a family $(p_\alpha)_{\alpha\in\Lambda}$ of Ambrose seminorms in the sense that $p_\alpha$ $(\alpha \in \Lambda)$ arises from a positive pseudo-inner product $<,>_\alpha$ such that the induced topology makes $A$ into a locally $m$-convex topological algebra.

Moreover, the following conditions are satisfied:
For any $x \in A$, there is an $x^* \in A$, such that

$$< xy, z >_\alpha = < y, x^* z >_\alpha$$
$$< yx, z >_\alpha = < y, zx^* >_\alpha$$

for any $y, z \in A$ and $\alpha \in \Lambda$.
Note that $x^*$ is not necessarily unique. But in the case that $A$ is proper, then $x^*$ is unique and $* : A \to A$, $x \mapsto x^*$ defines an involution in $A$.

# Some Results.

Let $(A, \{p_\alpha : p_\alpha \in \Lambda\})$ be a proper complete locally $m$-convex $H^*$-algebra.

Then $A$ can be turned into a locally pre-$C^*$-algebra (that is, not necessarily complete) defining the family $\{q_\alpha : \alpha \in \Lambda\}$ of $C^*$-seminorms on $A$ by

$$q_\alpha(x) = \sup\{p_\alpha(xy) : p_\alpha(y) \leq 1\},\ \alpha \in \Lambda,$$

so that

$$q_\alpha(x) \leq p_\alpha(x) \text{ for each } x \in A \text{ and } \alpha \in \Lambda.$$

Then the induced topology on $A$ is weaker than the given one.

Moreover,
$p_\alpha(xy) \leq q_\alpha(x)p_\alpha(y)$ for each $x, y \in A$ and $\alpha \in \Lambda$. [El Kinani, 2002].

# Our Results.

## Corollary

*Let* $(A, (p_\alpha)_{\alpha\in\Lambda})$ *be a complete locally m-convex* $H^*$*-algebra with continuous involution such that the respective projective system is perfect. Endow A with the locally m-convex topology previously defined and let* $\widetilde{A}$ *be its completion. Then:*

$$\mathcal{M}_l(\widetilde{A}) \cong \varprojlim \mathcal{M}_l(F_\alpha)$$

*within an isomorphism, where* $F_\alpha$, $\alpha \in \Lambda$ *are the normed factors in the Arens-Michael decomposition of* $\widetilde{A}$.

# Our Results.

**Proof.**

Since $*$ is an involution, $A$ is a proper algebra. By the comments preceeding the statement, $(A, (q_\alpha)_{\alpha\in\Lambda})$ is a locally pre-$C^*$-algebra. Hence its completion, $\widetilde{A}$, is a locally $C^*$-algebra; so the previous Theorem yields the assertion.We note that the factors of $\widetilde{A}$ are given by

$F_\alpha = (\widetilde{A}, \widetilde{q}_\alpha)/\ker \widetilde{q}_\alpha$, where $\widetilde{q}_\alpha$ is the extension of $q_\alpha$, $\alpha \in \Lambda$, to $\widetilde{A}$. □

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