Reversibility and Banach Algebras

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Definition

An element g of a group is called *reversible* if it is conjugate in the group to its inverse, i.e. there exists some map h, belonging to the group, such that the conjugate $g^h = h^{-1}gh$ equals g^{-1}. We say that h *reverses* g, in this case.

The Origins of the Concept:

- Classical conservative systems
  - Harmonic oscillator
  - n-body problem
- Billiards

How I became interested:

Approximation by $f(x) + g(y)$ on compacts in $\mathbb{R}^2$.

Approximation by $p(z^2, \bar{z}^2 + \bar{z}^3)$ on a disk.

Biholomorphuc classification of a pair of tangent real-analytic arcs.

Polynomial hull of a disk having an isolated complex tangent.

Each problem involves a pair of non-commuting involutions, so relates to a reversible element in some group of maps.

1. Notation

Let G be a group.

$I(G) := \{f \in G : f^2 = \mathrm{id}\}$. *(—involutions)*

$R_f := \{h \in G : f^h = f^{-1}\}$ (where $f^h = h^{-1}fh$). *(—reversers of f)*

$R(G) := \{f \in G : R_f \neq \emptyset\}$. *(—reversible elements)*

For $A \subset G$, denote $A^n = \{f_1 \cdots f_n : f_j \in A\}$. $A^\infty = \bigcup_{n=1}^\infty A_n$.

Elements of I^2 are called *strongly-reversible.* They are reversed by an involution.

Membership in I^n or R^n is a conjugacy invariant, and $I^2 \subset R$.

I^∞ and R^∞ are normal subgroups of G.

2. The Basic Questions

In each group, G, we ask:

- Which f are reversible in G?
- Which h reverse a given f?
- Describe I^∞.
- Describe R^∞.
- Is $I^n = I^\infty$ for some n?
- Is $R^n = R^\infty$ for some n?
- Does every nonempty R_g have an element of finite order? If so, what orders occur? Is $\min\{\mathrm{o}(h) : h \in R_g\}$ bounded, for $g \in R$?

If g is reversible by some element of finite order, then it is the product two elements of that (even) order. Thus results about R^n, combined with results about the order of reverses, also give information about factorizing elements of G as a product of elements of at most a given order.

3. Example: $\mathsf{GL}(n, \mathbb{C})$

Classification of linear reversible maps on $\mathbb{C}^n$ is simple. Suppose $F \in \mathsf{GL}(n, \mathbb{C})$ is reversible. Since the Jordan normal form of F^{-1} consists of blocks of the same size as F with reciprocal eigenvalues, the eigenvalues of F that are not ± 1 must split into groups of pairs $\lambda, 1/\lambda$. Furthermore, we must have the same number of Jordan blocks of each size for λ as for $1/\lambda$. Vice versa, if the eigenvalues of F are either ± 1 or split into groups of pairs $\lambda, 1/\lambda$ with the same number of Jordan blocks of each size, then both F and F^{-1} have the same Jordan normal form and are therefore conjugate to each other.

4. Survey of Known Results

G **abelian:**

$R = I = I^{\infty}$.

G **free:**

$R = I = \{1\}$.

Finite Groups

Dihedral D_n:

$I^2 = G = R$.

Symmetric S_n:

$I^2 = G = R$.

Finite Coxeter:

$I^2 = G = R$.

Alternating A_n:

$I^2 = R$.

$R \neq G$, except when $n \in \{1, 2, 5, 6, 10, 14\}$.

Quaternion 8-group:

$I^2 \neq R = G$.

Finite, simple G:

$R = \{1\}$ if $|G|$ is odd.

$G = R^2$, if $|G|$ is even, except for $\mathrm{PSU}(3, 3^2)$.

In general, $G \neq I^2$; When it happens is known.

Classical Groups

General Linear $\mathsf{GL}(n, F)$ **(**$n > 1$**):**

$I^2 = R.$
$I^4 = I^\infty = R^2.$

Special Linear $\mathsf{SL}(n, \mathbb{C})$**:**

$I^2 = R$ unless $n = 2 \pmod 4$
$R^2 = G.$

Orthogonal $\mathsf{O}(n, \mathbb{R})$

($\approx$ **spherical isometries):**

$I^2 = G.$

Special Orthogonal $\mathsf{SO}(n, \mathbb{R})$**:**

$I^2 = R.$
$I^3 = G$ if $n \geq 3.$
$I^2 = G$ if $n \neq 2(\mod 4).$

Unitary $\mathsf{U}(n, \mathbb{C})$**:**

$I^2 = R$
$I^4 = I^\infty.$

Special Unitary $\mathsf{SU}(n, \mathbb{C})$**:**

$I^2 \neq R.$
$I^3 \neq I^6 = G = R^2.$

Unitary Quaternionic $\mathsf{Sp}(n, \mathbb{C})$

$= \mathsf{Symp}(2n, \mathbb{C}) \cap \mathsf{U}(2n, \mathbb{C})$**:**

$I^2 \neq R = G = I^6.$

Spinor $\mathsf{Spin}(n, \mathbb{C})$**:**

$I^2 = G$ if $n = 0, 1, 7 \mod 8.$
$R = G$ unless $n = 2 \mod 4.$

$I^4 = G$ if $n \geq 5$.

Discrete Matrix Groups

$\mathsf{GL}(n,\mathbb{Z})$:

$I^{3n+9} = G$.

$I^2 \neq R \subset I^4$ when $n = 2$.

Modular $\mathsf{PSL}(2,\mathbb{Z})$**:**

$I^2 = R$.

Finite-dimensional Isometry Groups

Euclidean $\mathsf{Isom}(\mathbb{R}^n)$**:**

$I^2 = G$.

Orientation-preserving $\mathsf{Isom}^+(\mathbb{R}^n)$**:**

$I^3 = G$ if $n \geq 3$.

$I^2 = G$ if $n = 0$ or $3 \pmod 4$.

Hyperbolic $\mathsf{Isom}(\mathsf{H}^n)$**:**

$I^3 = G$ if $n \geq 2$.

Homeomorphism Groups

$\mathsf{Homeo}(\mathbb{R})$:

$I^2 \neq R$.
$I^3 \neq I^4 = G = R^2$.

$\mathsf{Homeo}^+(\mathbb{R})$:

$R^4 = G$.

$\mathsf{Homeo}(\mathbb{S}^1)$:

$I^2 \neq R$.
$I^3 = R^2 = G$.

$\mathsf{Homeo}^+(\mathbb{S}^1)$:

$I^2 \neq R$.
$I^3 = R^2 = G$.

$\mathsf{Homeo}(\mathbb{S}^n)$:

$G = I^6$ when $n = 2$ or 3. (Open for $n > 3$).

Compact surface MCG:

$I^n \neq G = I^\infty, \forall n \in \mathbb{N}, \quad$ if genus > 2.

Maps with extra Structure

Diffeomorphism $\mathsf{Diffeo}(\mathbb{R})$:

$I^2 \neq R$.

$I^3 \neq I^4 = G = R^2$.

$\mathsf{Diffeo}^+(\mathbb{R})$:

$R^4 = G$.

Formal germs on $(\mathbb{C}^n, 0)$:

$I^4 = I^\infty$ when $n = 1$.

$R^2 = R^\infty$ when $n = 1$.

$R^k = R^\infty$ with $k = 3 + 2 \cdot \text{ceiling}(\log_2 n)$ when $n \geq 2$.

$I^{15} = R^\infty$ when $n = 2$.

Piecewise-linear $\mathsf{PL}(\mathbb{R})$:

$I^2 \neq R$.

$I^3 \neq I^4 = G = R^2$.

$\mathsf{PL}^+(\mathbb{R})$:

$I = \{1\}$.

$R^4 = G$.

PL with finitely many nodes $\mathsf{PLF}(\mathbb{R})$:

$I^2 = R$.

$\mathsf{PLF}^+(\mathbb{R})$:

$R^4 = G$.

5. Banach Algebras

Let A be a Banach algebra. We may associate two collections of groups to A.

5.1. A^{-1} **and its subgroups.** Suppose A has identity (or adjoin one, if not) and $\|1\| = 1$.

Reversibility in A^{-1} is not interesting unless A is noncommutative. Also central reversibles are just central involutions, so the real problems are about the quotient

$$\frac{A^{-1}}{Z(A^{-1})} \equiv \mathsf{Inn}(A).$$

One interesting subgroup is

$$\mathsf{Iso}(A) = \{x \in A : \|x\| = \|x^{-1}\| = 1\}.$$

This coincides with the subgroup (denoted $\mathsf{U}(A)$ [1]) of unitary elements, in case A is a C^* algebra.

One may also focus on the connected component of 1 in either group, G, and on the intersection $G \cap E^c$ with the commutator of any subset $E \subset A$.

One also has the normal subgroup $\{x \in A : \|a - 1\| < 1\}^\infty$, which lies in the group $(\exp A)^\infty$.

For instance, Gustafson, Halmos, and Radjavi showed that for finite-dimensional (real or complex) Hilbert spaces H, the group $G = GL(H)$ has $I^4 = I^\infty$, and they noted that for infinite-dimensional H, we have $I^4 \neq I^7 = G(H)$. I don't know whether 7 is the best possible value in that statement. What happens with other C^* algebras?

It is known that for finite-dimensional $\mathsf{GL}(H)$, we have $I^2 = R$ in G, and also in the unitary subgroup. What about other C^* algebras?

5.2. **Aut**(A) **and its subgroups.** The main interesting subgroup (apart from the inner automorphism group, already mentioned, is the group of isometric isomorphims. This is often the same as $\mathrm{Aut}(A)$.

As an example, when X is a locally-compact Hausdorff space and $A = C_0(X, \mathbb{C})$, then $\mathrm{Aut}(A)$ is isomorphic to $\mathsf{Homeo}(X)$, so we have seen answers in case $X = \mathbb{R}^1$ and $X = \mathbb{S}^1$. What about noncommutative algebras?

References

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