On problems of Ghahramani–Lau and Johnson

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Arens products: Algebraic description

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...and the other way around:

$$\begin{aligned} \langle X \triangle Y, f \rangle &= \langle Y, f \triangle X \rangle \\ \langle f \triangle X, a \rangle &= \langle X, a \triangle f \rangle \\ \langle a \triangle f, b \rangle &= \langle f, b \cdot a \rangle \end{aligned}$$

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$\rightsquigarrow$ How to measure the degree of non-regularity?

Topological centres

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Definition (Dales–Lau '05)

$\mathcal{A}$ Strongly Arens Irregular (SAI) $:\Leftrightarrow \quad Z_\ell = Z_r = \mathcal{A}$

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$L_1(G)$ is SAI for any locally compact group G.

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Theorem (N '05)

The conjecture holds for all non-compact groups G s.t. $|G|$ non-measurable, OR $\kappa(G) \geq 2^{\chi(G)}$

One cannot prove in ZFC the existence of measurable cardinals (Ulam '30).

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Key technique (N): Factorization

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$\rightsquigarrow$ Distinction between G metrizable and non-metrizable

Commercial Break 1

For further structural results on $\mathbf{M}(G)^{**}$:

H.G. Dales, A.T.-M. Lau & D. Strauss

Second duals of measure algebras

Dissertationes Mathematicae (2011)

Central concepts: Thinness & Factorization

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Definition (L-N-P-S)

Let τ be a cardinal. Then $\mu \in \mathbf{M}(G)$ is τ-thin if $\exists P \subseteq G$ s.t. $|P| = \tau$ and $\mu * p \perp \mu * p' \quad \forall\ p \neq p'$ in P.

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Definition (N)

Let M be a subspace of $\mathbf{M}(G)$. Then M^* admits factorization if $\exists h \in \mathbf{B_1}(M^*)$ s.t.

$$\mathbf{B_1}(M^*) = \overline{\delta_G}\Box h \ .$$

Strategy of Proof: Thinness ⇒ Factorization ⇒ SAI

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Idea:

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We use decomposition

$$\mathbf{M}(G)^{**} = L_1(G)^{**} \oplus \mathbf{M}_s(G)^{**}$$

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For non-metrizable G we need refinement of decomposition

$\rightsquigarrow$ for compact subgroup $K \subseteq G$ consider right K-periodic measures on G ($\mu * k = \mu \ \ \forall\, k \in K$):

$$\mathbf{M}(G/K) = \mathbf{M}(G) * \lambda_K \text{ left ideal in } \mathbf{M}(G)$$

Factorization ⇒ Small Topological Centre

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Lemma

Let M be a Banach algebra, $M = M_0 \oplus M_1$. Assume $\exists h \in M^*$ s.t. $\mathbf{B_1}(M^{**}) \Box h = \mathbf{B_1}(M_0^*)$. Then $Z_\ell(M^{**}) \subseteq M_0 \oplus M_1^{**}$.

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Let $m \in Z_\ell(M^{**})$. Then $\psi_h : M^{**} \ni n \mapsto \langle m\Box n, h\rangle$ is w^*-cont.

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But $\varphi_h : n \mapsto n\Box h$ is w^*-cont. and $\psi_h = m \circ \varphi_h$.

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Since $\mathbf{B_1}(M^{**})$ is w^*-compact, m is w^*-cont. on

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so $m\,|_{M_0^*} \in M_0$. □

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$\rightsquigarrow$ Refined version for subspaces $M_2 = M_0 \oplus M_1$ in M

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Lemma (L-N-P-S)

Let M_0 subspace of $\mathbf{M}(G)$ s.t. if $\mu \in M_0$ then $|\mu| \in M_0$ and μ is τ-thin ($\tau > \aleph_0$).

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A direct sum $M_2 = M_0 \oplus M_1$ in $\mathbf{M}(G)$ is called *G-invariant* if $M_k * G \subseteq M_k$.

Factorization for thin measures, I

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Let $M_2 = M_0 \oplus M_1$ in $\mathbf{M}(G)$ be G-invariant.
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$\mathcal{O} \ni U = \{f \in \mathbf{B_1}(M_0^*) \mid |\langle f, \mu\rangle - \langle g_U, \mu\rangle| < \varepsilon_U \text{ for all } \mu \in F_U\}$
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Then $\exists h \in \mathbf{B_1}(\mathbf{M}(G)^*)$ that agrees with h_U on $F_U * x_U$ and vanishes on M_1. We check that $(\delta_{x_U} \Box h) |_{M_0} \in U$. □

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Theorem (L-N-P-S)

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The next result was proved by Prokaj ('03) for $G = \mathbb{R}$.

Theorem: Thinness of Singular Measures (L-N-P-S)

Let G *be non-discrete. Then every* $\mu \in \mathbf{M}_s(G)$ *is* $\mathfrak{c}\kappa(G)$*-thin.*

Proof of Main Theorem — metrizable case

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We conclude by using that $L_1(G)$ is SAI. □

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Further, for $\tau > \aleph_0$, put

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Corollary

K *compact subgroup. Then any* $\mu \in \mathbf{M}_{\mathrm{ss}}(G/K)$ *is* $|G/K|$*-thin.*

Characterizations of our classes of measures

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For compact subgroup K of G with G/K non-metrizable, put

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Lemma (L-N-P-S)

Let $\nu \in \mathbf{M}(G/K)$. Then:

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So it is enough to consider $m \in Z_\ell(\mathbf{M}(G)^{**}) \cap \mathbf{M}_{\mathbf{ai}}(G/K)^{**}$.

Proof of Main Theorem, II

By our inductive assumption, $m\Box\lambda_L \in \mathbf{M}(G/L)$ for all $L \in \mathcal{K}_\tau^\circ$.
Let $\mu \in \mathbf{M}(G/K)$ be the restriction of m to $C_0(G/K) \subseteq \mathbf{M}(G/K)^*$.
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Let $\delta \in \mathbf{M}_{\mathbf{ai}}(G/K)^{**}$ be a w^*-cluster point of $(\lambda_L)_{L\in\mathcal{K}_K}$. By approximate invariance, $(\lambda_L)_{L\in\mathcal{K}_K}$ is a BRAI for $\mathbf{M}_{\mathbf{ai}}(G/K)$. Since $m \in \mathbf{M}_{\mathbf{ai}}(G/K)^{**}$, we have

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One now obtains that $\mu \in \mathbf{M}_{\mathbf{ai}}$, and hence $m = \mu$. $\Box$

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Let G be compact metrizable, I a left ideal with BRAI in $L_1(G)$ (e.g., $L_1(G) * \lambda_K$). Then I is LSAI.

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Proof.

We have $I = L_1(G) * \mu$ for an idempotent measure μ. So I is WSC, has a sequential BRAI, and is an ideal in its bidual. We conclude by a result of Baker–Lau–Pym ('98). □

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Let G be any Polish group. Then $\mathbf{M}(G)$ is SAI.

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If G is a Polish, non locally compact group then every measure in $\mathbf{M}(G)$ is $\mathfrak{c}$-thin.

Factorization ideal of a Banach algebra

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Definition (N–P–S)

$\mathcal{A}$ Banach algebra. The left factorization ideal $\mathfrak{F}(\mathcal{A})$ is the ideal of subsets of $\mathbf{B_1}(\mathcal{A}^*)$ generated by $\{\mathbf{B_1}(\mathcal{A}^{**})\Box h \mid h \in \mathbf{B_1}(\mathcal{A}^*)\}$.

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$\mathfrak{F}(L_1(\mathbb{C})) \geq \aleph_1$ *and* $\mathfrak{F}(c_0)) = \mathfrak{d}_1 \geq \aleph_1$

Here, dominating number $\mathfrak{d}_1$ = least cardinal of $\mathcal{D} \subseteq \mathcal{L}_1 := \{f \in [0,1)^{\mathbb{N}} \mid \|f\|_1 \leq 1\}$ s.t. $\forall f \in \mathcal{L}_1\ \exists d \in \mathcal{D}$ with $f \leq d$

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$\boxed{L_1(\mathbb{G})}$ Banach algebra via $f * g = \Gamma_*(f \otimes g)$

Algebras and spaces over quantum groups

$L_\infty(\mathbb{G}) := M \quad L_1(\mathbb{G}) := M_* \quad L_2(\mathbb{G}) := L_2(M, \lambda)$

$\boxed{L_1(\mathbb{G})}$ Banach algebra via $f * g = \Gamma_*(f \otimes g)$

$$C_0(\mathbb{G}) \quad := \quad \overline{\{ (\mathrm{id} \otimes \tau)(W) \mid \tau \in \mathcal{T}(L_2(\mathbb{G})) \}}^{\|\cdot\|}$$

where W left fundamental unitary: $\Gamma(x) = W^*(1 \otimes x)W$

$\rightsquigarrow$ quantum measure algebra $\mathbf{M}(\mathbb{G}) = C_0(\mathbb{G})^*$

Uniform continuity

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$$\|\ell_x f - f\|_\infty < \varepsilon \quad \forall x \in U$$

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Recall: $(f_\alpha) \subseteq \mathbf{B_1}(\mathrm{LUC}(\mathbb{G}))$ is equi-LUC if $\forall \varepsilon > 0\ \exists U \in \mathfrak{U}(e)$ s.t.

$$\|\ell_x f_\alpha - f_\alpha\|_\infty < \varepsilon \quad \forall x \in U\ \forall \alpha$$

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- *(f_α) is equi-LUC*
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General result: $\mathcal{A}$ Banach algebra with BAI for action on Banach A-module X; if $K \subseteq X$ norm-compact, then $\exists a \in \mathcal{A}$ s.t. $K \subseteq X * a$.
Apply this with $\mathcal{A} = L_1(G)$, $X = \mathrm{LUC}(G)$, $K = \overline{\{f_\alpha\}}$. □

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$\mathbb{G}$ co-amenable LC quantum group. Then $\mathrm{U}(\mathbb{G}) = \mathbf{M}(\mathbb{G})$.

Commercial Break 2

J. Pachl

Uniform Spaces and Measures

Fields Institute Monographs (2012)

Some history

Problem (Johnson '72)

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- *(Argyros–Haydon '09) $\exists E$ s.t. $\mathcal{B}(E)$ amenable*

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Then $\mathrm{Ann}(X)$ is a complemented left ideal, so (Helemskiĭ '84) it has a BRAI e_α. Let $F \in \mathrm{Ann}(X)^{**}$ be a weak*-cluster point of e_α. Then F is a right identity in $\mathrm{Ann}(X)^{**}$ (w.r.t. the left Arens product).

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Put $E := \kappa^*(F) \in B(X^{**})$. For all $T \in \mathrm{Ann}(X)$:

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Thus, $E^*(m) = m$ for all $m \in X^\perp$.

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So, X is complemented in X^{**} – contradiction. $\square$

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$\mathcal{M}$ atomic $\Rightarrow$ $\mathcal{M} = \mathcal{A}^{**}$ with $\mathcal{A} = c_0 - \sum_{i \in I} \mathcal{K}(H_i)$ for Hilbert spaces H_i. Now Taylor's non-commutative Phillips theorem (1972) yields that $\mathcal{A}$ is not complemented in $\mathcal{A}^{**}$. □

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Let $\mathcal{M}$ be an inf.-dim. atomic vN algebra ($=$ direct sum of type I factors). Then $\mathcal{B}(\mathcal{M})$ is non-amenable.

Proof.

$\mathcal{M}$ atomic $\Rightarrow \mathcal{M} = \mathcal{A}^{**}$ with $\mathcal{A} = c_0 - \sum_{i \in I} \mathcal{K}(H_i)$ for Hilbert spaces H_i. Now Taylor's non-commutative Phillips theorem (1972) yields that $\mathcal{A}$ is not complemented in $\mathcal{A}^{**}$. □

Corollary

$\mathcal{B}(\mathcal{M})$ is non-amenable in the following cases:

- *$\mathcal{M} = \ell_\infty(I)$ for infinite I*
- *$\mathcal{M} = L_\infty(\mathbb{G})$ for any inf. discrete quantum group $\mathbb{G}$, in particular $\mathcal{M} = VN(G)$ for G compact*
- *$\mathcal{M} = \mathcal{B}(H)$ for any inf.-dim. Hilbert space H*

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NB: Our results also hold in the category of operator spaces and cb maps.

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If $\mathcal{M}_*$ is separable, $\mathcal{B}^s(\mathcal{M})$ is complemented left ideal in $\mathcal{B}(\mathcal{M})$.

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M_* separable $\Rightarrow$ WCG. Recall: WCG dual spaces have RNP. WCG and RNP are properties stable under isomorphism, so we get that $\mathcal{M}_*$ has the RNP. This is equivalent to $\mathcal{M}$ being atomic (Chu '81). Our earlier theorem – $\mathcal{B}(\mathcal{M})$ not amenable for atomic $\mathcal{M}$ – now gives a contradiction. □