LINEAR PRESERVER PROBLEMS: generalized inverse

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I. Introduction

Linear preserver problems is an active research area in Matrix, Operator Theory and Banach Algebras. These problems involve certain linear transformations on spaces of matrices, operators or Banach algebras... that will be designates by $\mathcal{A}$ in what follows. First, we start by listing the type of problems that are related to Linear Preserved Problems topic.

Problem I. Let F be a (scalar-valued, vector-valued, or set-valued) given function on $\mathcal{A}$. We would like to characterize those linear transformations ϕ on $\mathcal{A}$ which satisty

$$F(\phi(x)) = F(x) \qquad (x \in \mathcal{A}).$$

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Examples : (i) scalar-valued : $\mathcal{A} = M_n(C)$ and $F(x) = det(x)$

Theorem [G. Frobenius 1897]

A linear map ϕ from $\mathcal{M}_n(\mathbb{C})$ into $\mathcal{M}_n(\mathbb{C})$ preserves the determinant (i.e. $\det(\phi(x)) = \det(x)$) if and only if it takes one of the following forms

$$\phi(x) = axb \quad or \quad \phi(x) = ax^{tr}b, \qquad for\ all\ x \in \mathcal{M}_n(\mathbb{C})$$

where $a, b \in \mathcal{M}_n(\mathbb{C})$ are non-singular matrices such that $\det(ab) = 1$, and x^{tr} stands for the transpose of x.

Problem I

(ii) $\mathcal{A} = C(K)$ and $F(X) = ||X||$ (scalar-valued).
Let K be a compact metric space and $C(K)$ the Banach space of continuos real valued functions defined on K, with the supremun norm.

Theorem [Banach 1932]

Let $\phi : C(K) \to C(K)$ be a surjective linear map. Then ϕ is an isometry (i.e. $||\phi(f)||_\infty = ||f||_\infty$) if and only if

$$\phi(f(t)) = h(t)f(\varphi(t)), \quad t \in K$$

where $|h(t)| = 1$ and φ is a homeomorphism of K onto itself.

Problem I

Kadison (1951) described the surjective linear isometries of C^*-algebras.

Theorem [Kadison]

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective linear map. Then
ϕ is an isometry if and only if there is unitary $u \in \mathcal{B}$ and a C^*-isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\phi = u\varphi$.

Problem I

(iii) vector-valued : $\mathcal{A}$ C^*-algebra and $F : \mathcal{A} \to \mathcal{A}$, $F(X) = |X|$, the absolute value of X (i.e. $|X| = (X^*X)^{1/2}$).

Theorem

Let $\mathcal{A}$ be a C^*-algebra and $\phi : \mathcal{A} \to \mathcal{A}$ be a surjective linear map. Then $|\phi(x)| = |x|$ for all $x \in \mathcal{A}$ if and only if there exists an unitary element $u \in \mathcal{A}$ such that $\phi(x) = ux$ for all $x \in \mathcal{A}$.

(iv) set-valued : $\mathcal{A}$ Banach algebra and $F(X) = \sigma(X)$.

Problem II. Let S be a given subset of $\mathcal{A}$. We would like to characterize those linear map ϕ on $\mathcal{A}$ wich satisfy :

$$X \in S \Longrightarrow \phi(X) \in S \quad (X \in \mathcal{A})$$

or satisfy

$$X \in S \iff \phi(X) \in S \quad (X \in \mathcal{A}).$$

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Examples :
(i) S the set of idempotent, nilpotent, Fredholm, semi-Fredholm operators.
(ii) $S = \mathcal{A}^{-1} = \{X \in \mathcal{A}; X \text{ invertible}\}$
(iii) $S = \{X \in \mathcal{A}; \exists Y \in \mathcal{A}; XYX = X\}$, the set of all the elements of $\mathcal{A}$ having a generalized inverse.
The two previous examples, where the first one is related to Kaplansky problem, will be developed in detail in this presentation, later on.

Problem III

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$$X \,\mathcal{R}\, Y \Longrightarrow \phi(X) \,\mathcal{R}\, \phi(Y), \quad (X, Y \in \mathcal{A})$$

or satisfy

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Examples :
(i) $\mathcal{R}$ could be commutativity
(ii) $\mathcal{R}$ could be similarity
(iii) "zero product preserving mappings" i.e.
$XY = 0 \iff \phi(X)\phi(Y) = 0.$

Problem IV

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$$F(\phi(X)) = \phi(F(X)) \quad (X \in \mathcal{A}).$$

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Examples :
(i) $\mathcal{A} = M_n(C)$ and $F(X) = X^\dagger$
($X^\dagger$ stands for the Moore-Penrose inverses of X).

(ii) $F(X) = |X|$, the absolute value of X (i.e. $|X| = (X^*X)^{1/2}$).

(iii) $F(X) = X^n$, n being a fixed integer not less than 2. When $n = 2$, ϕ is a Jordan homomorphism.

Jordan homomorphism

A linear map $\phi : \mathcal{A} \rightarrow \mathcal{B}$ between two unital complex Banach algebras is called *Jordan homomorphism* if $\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a)$ for all a, b in $\mathcal{A}$,

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or equivalently, $\phi(a^2) = \phi(a)^2$ for all a in $\mathcal{A}$.

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or equivalently, $\phi(a^2) = \phi(a)^2$ for all a in $\mathcal{A}$.
We say that ϕ is *unital* if $\phi(1) = 1$.

Remarks

(1) ϕ is homomorphism $\Rightarrow$ ϕ is Jordan homomorphism.
(2) ϕ is anti-homomorphism (i.e. $\phi(ab) = \phi(b)\phi(a)$) $\Rightarrow$ ϕ is Jordan homomorphism.
(3) The converse is true in some cases :
for instance if ϕ is onto and if $\mathcal{B}$ is prime, that is $a\mathcal{B}b = \{0\} \Rightarrow a = 0$ or $b = 0$.
(4) It is well-known that a unital Jordan homomorphism preserves invertibility,
(i.e. $x \in \mathcal{A}^{-1} \Rightarrow \phi(x) \in \mathcal{B}^{-1}$).

Kaplansky's problem

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Let ϕ *be a unital surjective linear map between two semi-simple Banach algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *which preserves invertibility (i.e.,* $\phi(x) \in \mathcal{B}^{-1}$ *whenever* $x \in \mathcal{A}^{-1}$*). Is it true that* ϕ *is a Jordan homomorphism ?*

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This problem has been first solved in the finite-dimensional case:

Theorem [J. Dieudonné, 1943]

Let $\phi : M_n(C) \to M_n(C)$ be a surjective, unital linear map. If $\sigma(\phi(x)) \subseteq \sigma(x)$ for all $x \in M_n(C)$, then there is an invertible element $a \in M_n(C)$ such that ϕ takes one of the following forms:

$$\phi(x) = axa^{-1} \quad or \quad \phi(x) = ax^{tr}a^{-1}.$$

Kaplansky's problem

In the commutative case the well-known Gleason-Kahane-Żelazko theorem provides an affirmative answer.

Theorem [Gleason-Kahan-Zelazko, 1968]

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ a semi-simple commutative Banach algebra.
If $\phi : \mathcal{A} \to \mathcal{B}$ is a linear map such that $\sigma(\phi(x)) \subseteq \sigma(x)$ for all $x \in \mathcal{A}$, then ϕ is an homomorphism.

Kaplansky's problem

In the non-commutative case, the best known results so far are due to Aupetit and Sourour.

Theorem [Sourour, 1996]

Let X, Y be two Banach spaces and let $\phi : B(X) \to B(Y)$ a linear, bijective and unital map. Then the following properties are equivalent:
(i) $\sigma(\phi(T)) \subseteq \sigma(T)$ for all $T \in B(X)$;
(ii) ϕ is an isomorphism or an anti-isomorphism;
(iii) either there is $A : Y \to X$ invertible such that $\phi(T) = A^{-1}TA$ for all $T \in B(X)$;
or, there is $B : Y \to X^*$ invertible such that $\phi(T) = B^{-1}T^*B$ for all $T \in B(X)$, and in this case X and Y are reflexive.

Kaplansky's problem

We will say that a C^*-algebra $\mathcal{A}$ is *of real rank zero* if the set formed by all the real linear combinations of (orthogonal) projections is dense in the set of self-adjoint elements of $\mathcal{A}$.
It is well known that this property is satisfied by every von Neumann algebra, and in particular by the C^*-algebra $B(H)$ of all bounded linear operators on a Hilbert space H, and by the Calkin algebra $C(H) = B(H)/K(H)$.

Theorem [B. Aupetit, 2000]

Let $\mathcal{A}$ be a C^*-algebra of real rank zero and $\mathcal{B}$ a semi-simple Banach algebra.
If $\phi : \mathcal{A} \to \mathcal{B}$ is a surjective linear map such that $\sigma(\phi(x)) = \sigma(x)$ for all $x \in \mathcal{A}$, then ϕ is a Jordan isomorphism.

Kaplansky's problem

Remark

The Kaplansky problem is still open when $\mathcal{A}, \mathcal{B}$ are C^*-algebras.

Remark Many other linear preserver problems, like the problem of characterizing linear maps preserving idempotents or nilpotents or commutativity,... that were first solved for matrix algebras, have been recently extended to the infinite-dimensional case (B. Aupetit; Brešar; M. Mathien, L. Molnar; L. Rodman; Šemrl; A.R. Sourour;...).

II. Generalized inverse preservers maps

Definition An element $b \in \mathcal{A}$ is called a *generalized inverse* of $a \in \mathcal{A}$ if b satisfies the following two identities

$$aba = a \quad \text{and} \quad bab = b.$$

Let $\mathcal{A}^{\wedge}$ denote the set of all the elements of $\mathcal{A}$ having a generalized inverse.

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Theorem [Kaplansky, 1948]

Let $\mathcal{A}$ be a Banach algebra:
- *If $\mathcal{A} = \mathcal{A}^\wedge$ then $\mathcal{A}$ is finite dimensional.*
- *Furthermore, if $\mathcal{A}$ is semi-simple, then*
$\mathcal{A} = \mathcal{A}^\wedge \iff dim(A) < \infty.$

Generalized inverse

Definition We say that $\phi : \mathcal{A} \to \mathcal{B}$ *preserves generalized invertibility in both directions* if $\quad x \in \mathcal{A}^\wedge \iff \phi(x) \in \mathcal{B}^\wedge.$

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Remark Observe that, every $n \times n$ complex matrix has a generalized inverse, and therefore, every map on a matrix algebra preserves generalized invertibility in both directions. So, we have here an example of a linear preserver problem which makes sense only in the infinite-dimensional case.

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Let H be an infinite-dimensional separable complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H and $K(H) \subset B(H)$ be the closed ideal of all compact operators.

We denote the Calkin algebra $B(H)/K(H)$ by $C(H)$. Let $\pi : B(H) \to C(H)$ be the quotient map.

Generalized inverse

Theorem [M. Mbekhta, L. Rodman and P. Šemrl, 2006]

Let H be an infinite-dimensional separable Hilbert space and let $\phi : B(H) \to B(H)$ be a bijective continuous unital linear map preserving generalized invertibility in both directions. Then

$$\phi(K(H)) = K(H),$$

and the induced map $\varphi : C(H) \to C(H)$, (i.e. $\varphi \circ \pi = \pi \circ \phi$), is either an automorphism, or an anti-automorphism.

Generalized inverse

Let $F(H) \subset B(H)$ the ideal of all finite rank operators,
We say that $\phi : B(H) \to B(H)$ is surjective up to finite rank (resp. compact) operators if for every $A \in B(H)$ there exists $B \in B(H)$ such that $A - \phi(B) \in F(H)$ (resp. $K(H)$).

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Theorem [M.Mbekhta and P. Šemrl, 2009]

Let H be an infinite-dimensional separable Hilbert space and $\phi : B(H) \to B(H)$ a surjective up to finite rank operators linear map. If ϕ preserves generalized invertibility in both directions, then

$$\phi(K(H)) \subseteq K(H)$$

and the induced map $\varphi : C(H) \to C(H)$ is either an automorphism, or an anti-automorphism multiplied by an invertible element $a \in C(H)$.

III. Semi-Fredholm preserver maps

We recall that an operator $A \in B(H)$ is said to be *Fredholm* if its range is closed and both its kernel and cokernel are finite-dimensional, and is *semi-Fredholm* if its range is closed and its kernel or its cokernel is finite-dimensional.

We denote by $SF(H) \subset B(H)$ the subset of all semi-Fredholm operators.

Semi-Fredholm

The above theorem allows us to establish the following result which is of independent interest.

We say that a map $\phi : B(H) \to B(H)$ *preserves semi-Fredholm operators in both directions* if for every $A \in B(H)$ the operator $\phi(A)$ is semi-Fredholm if and only if A is.

Theorem

Let H be an infinite-dimensional separable Hilbert space and $\phi : B(H) \to B(H)$ a surjective up to compact operators linear map. If ϕ preserves semi-Fredholm operators in both directions, then

$$\phi(K(H)) \subseteq K(H)$$

and the induced map $\varphi : C(H) \to C(H)$ is either an automorphism, or an anti-automorphism multiplied by an invertible element $\mathbf{a} \in C(H)$.

Semi-Fredholm

For a Fredholm operator $T \in B(H)$ we define the index of T by

$$\operatorname{ind}(T) = \dim \operatorname{Ker} T - \operatorname{codim} \operatorname{Im} T \in \mathbb{Z}.$$

Theorem

Under the same hypothesis and notation as in the above theorem, the following statements hold :
(i) ϕ preserves Fredholm operators in both directions;
(ii) there is an $n \in \mathbb{Z}$ such that either

$$\operatorname{ind}(\phi(T)) = n + \operatorname{ind}(T)$$

for every Fredholm operator T, or

$$\operatorname{ind}(\phi(T)) = n - \operatorname{ind}(T)$$

for every Fredholm operator T.

Semi-Fredholm

Lemma

Let $A \in B(H)$. *Then the following are equivalent:*
(i) A *is semi-Fredholm,*
(ii) for every $B \in B(H)$ *there exists* $\delta > 0$ *such that* $A + \lambda B \in B(H)^{\wedge}$ *for every complex* λ *with* $|\lambda| < \delta$.

Semi-Fredholm

Lemma

Let $A \in B(H)$. Then the following are equivalent:
(i) A is semi-Fredholm,
(ii) for every $B \in B(H)$ there exists $\delta > 0$ such that $A + \lambda B \in B(H)^\wedge$ for every complex λ with $|\lambda| < \delta$.

Corollary

Let H be an infinite-dimensional separable Hilbert space and $\phi : B(H) \to B(H)$ a surjective up to finite rank op linear map. Then ϕ preserves generalized invertibility in both directions implies ϕ preserves semi-Fredholm operators in both directions.

IV. Additive maps strongly preserving generalized inverses

We denote by $\mathcal{A}^{-1}$ the set of invertible elements of $\mathcal{A}$.
We shall say that an additive map $\phi : \mathcal{A} \to \mathcal{B}$ *strongly preserves invertibility* if $\phi(x^{-1}) = \phi(x)^{-1}$ for every $x \in \mathcal{A}^{-1}$.
Similarly, we shall say that ϕ *strongly preserves generalized invertibility* if $\phi(y)$ is a generalized inverse of $\phi(x)$ whenever y is a generalized inverse of x.

Remark. One easily checks that a Jordan homomorphism strongly preserves invertibility (resp. generalized inverses).

The motivation for this problem is Hua's theorem (1949) which states that

every unital additive map ϕ between two fields such that $\phi(x^{-1}) = \phi(x)^{-1}$ is an isomorphism or an anti-isomorphism.

strongly preserving generalized inverses

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every unital additive map ϕ between two fields such that $\phi(x^{-1}) = \phi(x)^{-1}$ is an isomorphism or an anti-isomorphism.

Theorem [N.Boudi and M.Mbekhta, 2010]

Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive map. Then ϕ strongly preserves invertibility if and only if $\phi(1)\phi$ is a unital Jordan homomorphism and $\phi(1)$ commutes with the range of ϕ.

For the special case of the complex matrix algebra $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$, we derive the following corollary that provides a more explicit form of linear maps strongly preserving invertibility.

Corollary

Let $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$, be a linear map. Then the following conditions are equivalent:
(1) ϕ preserves invertibility ;
(2) there is a $\lambda \in \{-1, 1\}$ such that ϕ takes one of the following forms:

$$\phi(x) = \lambda axa^{-1} \quad or \quad \phi(x) = \lambda ax^{tr}a^{-1},$$

for some invertible element $a \in \mathcal{M}_n(\mathbb{C})$.

Theorem [N.Boudi and M.Mbekhta]

Let $\mathcal{A}$ and $\mathcal{B}$ be unital complex Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive map such that $1 \in Im(\phi)$ or $\phi(1)$ is invertible. Then the following conditions are equivalent:
(i) ϕ strongly preserves generalized invertibility;
(ii) $\phi(1)\phi$ is a unital Jordan homomorphism and $\phi(1)$ commutes with the range of ϕ.

strongly preserving generalized inverses

For the special case of linear maps over the complex matrix algebra $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$, we deduce the following corollary that gives a more explicit form of the linear maps strongly preserving generalized invertibility.

Corollary

Let $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$, be a linear map. Then ϕ strongly preserves generalized inverses if and only if either $\phi = 0$ or there is $\lambda \in \{-1, 1\}$ such that ϕ takes one of the following forms:

$$\phi(x) = \lambda a x a^{-1} \quad or \quad \phi(x) = \lambda a x^{tr} a^{-1},$$

for some invertible element $a \in \mathcal{M}_n(\mathbb{C})$.

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N. Boudi and M.Mbekhta *Additive maps preserving strongly generalized inverses*, J. Operator Theory 64 (2010), 117-130

V. Moore-Penrose inverses preservers maps

In the context C^*-algebras, it is well known that every generalized invertible element a has a unique generalized inverse b for which ab and ba are projections, such an element b is called the *Moore-Penrose inverse* of a and denoted by $a^\dagger$.
In other words, $a^\dagger$ is the unique element of $\mathcal{A}$ that satisfies:

$$aa^\dagger a = a,\ a^\dagger aa^\dagger = a^\dagger,\ (aa^\dagger)^* = aa^\dagger,\ (a^\dagger a)^* = a^\dagger a.$$

Let $\mathcal{A}^\dagger$ denotes the set of all elements of $\mathcal{A}$ having a Moore-Penrose inverse.
We will say that a linear map $\phi : \mathcal{A} \to \mathcal{B}$ *preserves strongly Moore-Penrose invertibility* if

$$\phi(x^\dagger) = \phi(x)^\dagger, \quad \forall x \in \mathcal{A}^\dagger.$$

$\mathcal{A}$ and $\mathcal{B}$ will denote a C^*-algebras, and we will say that a linear map $\phi : \mathcal{A} \to \mathcal{B}$ is C^*-*Jordan homomorphism* if it is a Jordan homomorphism which preserves the adjoint operation, i.e. $\phi(x^*) = \phi(x)^*$ for all x in $\mathcal{A}$.
The C^*-homomorphism and C^*-anti-homomorphism are analogously defined.

Theorem

Let $\mathcal{A}$ be a C^*-algebra of real rank zero and $\mathcal{B}$ a prime C^*-algebra. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective, unital linear map. Then the following conditions are equivalent:
1) $\phi(x^\dagger) = \phi(x)^\dagger$ for all $x \in \mathcal{A}^\dagger$;
2) ϕ is either a C^*-homomorphism or a C^*-anti-homomorphism.

Moore-Penrose inverses

As an application of the above theorem in the context of the C^*-algebra $\mathcal{B}(H)$ of bounded linear operator on complex separable Hilbert space, we derive the following result which characterizes the surjective unital additive maps from $\mathcal{B}(H)$ onto itself that strongly preserves Moore-Penrose inverses.

Denote by $\mathcal{B}^\dagger(H)$ the set of the operators on H that possess a Moore-Penrose inverse.

Corollary

Let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective unital additive map. Then the following conditions are equivalent:

(1) $\phi(T^\dagger) = \phi(T)^\dagger$ for all $T \in \mathcal{B}^\dagger(H)$;

(2) there is a unitary operator U in $\mathcal{B}(H)$ such that ϕ takes one of the following forms

$$\phi(T) = UTU^* \quad or \quad \phi(T) = UT^{tr}U^* \quad for\ all\ T,$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed

In connection with Theorem, we conclude by the following conjecture

conjecture

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective linear map. Then the following conditions are equivalent:
1) $\phi(x^\dagger) = \phi(x)^\dagger$ for all $x \in \mathcal{A}^\dagger$;
2) ϕ is a C^*-Jordan homomorphism.

VI. Additive preservers of the ascent, descent and related subsets

The *ascent* $\mathrm{a}(T)$ and *descent* $\mathrm{d}(T)$ of $T \in \mathcal{L}(X)$ are defined by
$\mathrm{a}(T) = \inf\{n \geq 0\colon ker(T^n) = ker(T^{n+1})\}$
$\mathrm{d}(T) = \inf\{n \geq 0\colon R(T^n) = R(T^{n+1})\}$, where the infimum over the empty set is taken to be infinite.

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$\mathrm{d}(T) = \inf\{n \geq 0\colon R(T^n) = R(T^{n+1})\}$, where the infimum over the empty set is taken to be infinite.
An operator $T \in \mathcal{L}(X)$ is said to have a *Drazin inverse*, or to be *Drazin invertible*, if there exists $S \in \mathcal{L}(X)$ and a non-negative integer n such that

$$T^{n+1}S = T^n, STS = S \text{ and } TS = ST. \tag{1}$$

Note that if T possesses a Drazin inverse, then it is unique and the smallest non-negative integer n in (1) is called the *index* of T and is denoted by $\mathrm{i}(T)$. It is well known that T is Drazin invertibe if and only if it has finite ascent and descent, and in this case $\mathrm{a}(T) = \mathrm{d}(T) = \mathrm{i}(T)$.

Recall also that an operator $T \in \mathcal{L}(X)$ is called *upper* (resp. *lower*) *semi-Fredholm* if $\mathrm{R}(T)$ is closed and $\dim \mathrm{N}(T)$ (resp. codimR(T)) is finite. The set of such operators is denoted by $\mathcal{F}_+(X)$ (resp. $\mathcal{F}_-(X)$). The *Fredholm* and *semi-Fredholm* subsets are defined by $\mathcal{F}(X) := \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$ and $\mathcal{F}_\pm(X) := \mathcal{F}_+(X) \cup \mathcal{F}_-(X)$, respectively.

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Let $\mathcal{S}$ denotes one of the subsets (i)-(vii). A surjective additive maps $\Phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ is said to *preserve* $\mathcal{S}$ in the both direction if $T \in \mathcal{S}$ if and only if $\Phi(T) \in \mathcal{S}$.

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(v) there exists an invertible bounded linear, or conjugate linear, operator $A : H \to K$ and a non-zero complex number c such that $\Phi(S) = cASA^{-1}$ for all $S \in \mathcal{L}(H)$.

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