Spectrally bounded and spectrally isometric operators

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Nadia Boudi and Martin Mathieu,
Elementary operators that are spectrally bounded,
Operator Theory: Advances and Applications **212** (2011), 1–15.

Martin Mathieu and Ahmed R. Sourour,
Spectral isometries on non-simple C^*-algebras,
submitted to *Proc. Amer. Math. Soc.*

Notation:

A, B unital complex Banach algebras; $x \in A$

$$\sigma(x) = \{\lambda \in \mathbb{C} \mid \lambda - x \text{ not invertible}\}$$

the *spectrum* of x, and

$$r(x) = \sup\{|\lambda| \mid \lambda \in \sigma(x)\} = \lim_{n\to\infty} \|x^n\|^{1/n}$$

the *spectral radius* of x.

Definition

$T: A \to B$ linear is

- spectrally bounded if $\exists\, M > 0\ \forall x \in A:\ r(Tx) \le M\, r(x)$;
- a spectral isometry if $r(Tx) = r(x) \quad \forall x \in A$.

Examples:

- T unital (i.e., $T1_A = 1_B$), invertibility preserving
  $\Rightarrow$ T spectrally bounded;
- T spectrum preserving (i.e., $\sigma(Tx) = \sigma(x)$ for all $x \in A$)
  $\Rightarrow$ T spectral isometry;
- T Jordan epimorphism (i.e., $T(x^2) = (Tx)^2$ for all $x \in A$)
  $\Rightarrow$ T invertibility preserving and unital;
- T Jordan isomorphism (i.e., bijective Jordan homomorphism)
  $\Rightarrow$ T spectrum preserving.
- A uniform algebra, T bounded $\Rightarrow$ T spectrally bounded;
- $A = M_n(\mathbb{C})$, $B = \mathbb{C}$, T normalised trace $\Rightarrow$ T spectrally bounded.

Theorem (Pták, 1978)

The following two conditions are equivalent:

(a) $L_a \colon x \mapsto ax$ *is spectrally bounded on* A;

(b) $a \in \mathcal{Z}(A)$, *where* $\mathcal{Z}(A)$ *is the centre modulo the radical.*

Theorem (Curto–Mathieu, 1995)

The following two conditions on $a, b \in A$ *are equivalent:*

(a) *The generalised inner derivation* $L_a - R_b \colon x \mapsto ax - xb$ *is spectrally bounded on* A;

(b) $a \in \mathcal{Z}(A)$ *and* $b \in \mathcal{Z}(A)$.

[R. Curto and M. Mathieu, *Spectrally bounded generalized inner derivations*, Proc. Amer. Math. Soc. **123** (1995), 2431–2434.]

Elementary operators

for $a, b \in A$ let $M_{a,b}\colon x \mapsto axb,\ x \in A$, the two-sided multiplication;

$$S = \sum_{j=1}^{n} M_{a_j,b_j}$$

with $a_1, \dots, a_n,\ b_1, \dots, b_n \in A$ is called an elementary operator on A, and $\mathcal{E}\ell(A)$ is the algebra of all those operators;

if $S = M_{a,b} + M_{c,d}$ we say S *has length (at most) two*.

Elementary operators that are spectrally bounded

Theorem (Boudi–Mathieu, 2011)

Let A be a unital Banach algebra.
Then $S = M_{a,b} + M_{c,d} \in \mathcal{E}\ell(A)$ is spectrally bounded if and only if, for every primitive ideal P of A, there exists $\beta_P \in \mathbb{C}$ such that

$$\pi_P((b + \beta_P d)a) \in \mathbb{C} \quad \textit{and} \quad \pi_P(d(c - \beta_P a)) \in \mathbb{C}$$

and either $\pi_P(b + \beta_P d)\pi_P(c - \beta_P a) = 0$ or $\beta_P = 0$ and $\pi_P(da) = 0$. In particular, $ba + dc \in \mathcal{Z}(A)$ in this case.

Notation: for $P \in Prim(A)$, (π_P, E_P) irreducible representation with $\ker \pi_P = P$. **Pták**: $M_{a,b}$ spectrally bounded $\iff$ $ba \in \mathcal{Z}(A)$ (as $r(axb) = r(bax)$ for all x).

Open Problem:

Characterize general spectrally bounded elementary operators!

General comments

Suppose $T\colon A \to B$ is a spectral isometry between two unital Banach algebras A and B.

- The surjectivity assumption is inevitable, as is well known.
- If A is semisimple, then T is injective [Mathieu–Schick, 2002] and the inverse T^{-1} is a bijective spectral isometry as well.
- In this case, the codomain is semisimple too (see Lemma below) and it follows that T is bounded (Aupetit's Lemma). The open mapping theorem entails that T^{-1} is bounded as well.
- When T is non-unital and A and B are C^*-algebras, then $T1$ is a central unitary; thus, replacing T by $\tilde{T}$ defined by $\tilde{T}(x) = (T1)^{-1}Tx$, $x \in A$ we can reduce the general to the unital case [Lin–Mathieu, 2007].

Lemma 1

Let A and B be unital Banach algebras. Let $T\colon A \longrightarrow B$ be a surjective spectral isometry. Then $T\,\mathrm{rad}(A) = \mathrm{rad}(B)$.

Lemma 2

Let $T\colon A \longrightarrow B$ be a unital surjective spectral isometry from the semisimple unital Banach algebra A onto the unital Banach algebra B. If A is commutative then B is commutative.

Lemma 3

Let $S\colon B \to B$ be a unital surjective spectral isometry on a unital Banach algebra B. Let I be a closed ideal of B such that the quotient algebra B/I is semisimple. If $r(Sy + I) = r(y + I)$ for all $y \in B$ then S induces a unital surjective spectral isometry $S^I\colon B/I \to B/I$ such that $S^I(y + I) = Sy + I$ for all $y \in B$.

Lemma 4

Let A and B be unital semisimple Banach algebras, and let $T\colon A \to B$ be a unital surjective spectral isometry. Let I be a closed ideal of B such that B/I is semisimple and that each unital surjective spectral isometry $S\colon B/I \to B/I$ is multiplicative or anti-multiplicative. Let $a \in A$ and put $A_0 = \{a\}^{cc}$. Let $B_0 = TA_0$. For all $b_1, b_2 \in B_0$ and $x \in \{a\}^c$, we have

$$T^{-1}(b_1 b_2)\, x + I = x\, T^{-1}(b_1 b_2) + I. \tag{1}$$

Here, $X^c = \{y \in A \mid yx = xy \text{ for all } x \in X\}$ denotes the commutant of $X \subseteq A$.

Restrictions to commutative subalgebras

Proposition

Let A be a unital C-algebra and let B be a unital Banach algebra. The following conditions on a unital surjective spectral isometry $T\colon A \to B$ are equivalent.*

(a) *T is a Jordan isomorphism;*

(b) *TA_0 is a subalgebra of B for every commutative unital subalgebra A_0 of A;*

(c) *$T\big(\{a\}^{cc}\big)$ is a subalgebra of B for every element $a \in A_{sa}$;*

(d) *$T\,C^*(a)$ is a subalgebra of B for every element $a \in A_{sa}$.*

Theorem (Mathieu–Sourour, 2011)

Let A and B be unital semisimple Banach algebras, and let $T\colon A \to B$ be a unital surjective spectral isometry. Suppose that B has a separating family $\mathscr{I}$ of closed ideals I such that B/I is semisimple and that each unital surjective spectral isometry $S^I\colon B/I \to B/I$ is multiplicative or anti-multiplicative. Then T preserves invertibility. If, moreover, A is a C-algebra, then T is a Jordan isomorphism.*

generalises [Costara–Repovš, 2010] for sufficiently many finite-dimensional representations

Spectral isometries on non-simple C^*-algebras

Corollary

Let $T\colon A \longrightarrow B$ be a unital surjective spectral isometry from a separable unital C^*-algebra with Hausdorff spectrum onto a unital C^*-algebra B. Then T is a Jordan isomorphism.

Corollary

Let A and B be unital C^*-algebras, and let $T\colon A \longrightarrow B$ be a unital surjective spectral isometry. Suppose that A has real rank zero and no tracial states and that $Prim(A)$ contains a dense subsets of closed points. Then T is a Jordan isomorphism.

[M. Mathieu, *Spectrally bounded operators on simple C^*-algebras,* II, Irish Math. Soc. Bull. **54** (2004), 33–40.]