The (continuous) Fourier algebra of a locally compact groupoid

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Outline

1. Groupoids: definition, notation and examples.

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2. Fourier-Stieltjes and Fourier algebras.

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2. Fourier-Stieltjes and Fourier algebras.
3. A decomposition of the Fourier algebra.

Groupoid: definition

Groupoids

A *groupoid* is a set G together with a subset $G^2 \subseteq G \times G$, an associative product $\left\{ \begin{array}{ccc} G^2 & \to & G \\ (\delta, \gamma) & \to & \delta\gamma \end{array} \right.$ and an inverse $\left\{ \begin{array}{ccc} G & \to & G \\ \gamma & \to & \gamma^{-1} \end{array} \right.$ such that:

$(\gamma^{-1}, \gamma) \in G^2, \forall \gamma \in G$ and if $(\delta, \gamma) \in G^2$ then $\delta^{-1}\delta\gamma = \gamma$, $\delta\gamma\gamma^{-1} = \delta$.

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$$\begin{tikzcd} \bullet \arrow[r, bend left, "\gamma"] & \bullet \arrow[l, bend left, "\gamma^{-1}"] \arrow[r, bend left, "\delta"] & \bullet \end{tikzcd}$$

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$$\bullet \xrightarrow{\delta} \bullet$$

Notation

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$G^0 = \{\gamma^{-1}\gamma : \gamma \in G\}$ are the *units* of G.

The range map is $r : G \to G^0$, $r(\gamma) = \gamma\gamma^{-1}$,
The source map is $s : G \to G^0$, $s(\gamma) = \gamma^{-1}\gamma$.

If $u \in G^0$, $G^u = r^{-1}(u)$, $G_u = s^{-1}(u)$ and $G_u^u = G^u \cap G_u$ (the *isotropy group at* u).

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G is a *locally compact groupoid* if G, G^0 are l.c. Hausdorff 2^{nd} countable spaces and the operations are continuous.

Examples

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Groups are groupoids with $G^0 = \{e\}$.

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Equivalence relations

If R is an equivalence relation on a set U, it is a groupoid $G_R \subseteq U \times U$ with the following structure:

- $\big((w, v'), (v, u)\big) \in G^2 \Leftrightarrow v = v'$,
  $(w, v)(v, u) = (w, u)$.
- $(v, u)^{-1} = (u, v)$,
- $r(v, u) = (v, v) \simeq v$, $s(v, u) = (u, u) \simeq u$.

More examples

Finite case

Let G be a **finite** and **transitive** groupoid ($G_v^u \neq \phi$).

- Then $G(u) \cong G(v)$, $\forall u, v \in G^0$.
- Fix $u \in G^0$. Then $G \cong G^0 \times G_u^u \times G^0$: define $\nu : G^0 \to G^u$, $\nu(v) \in G_v^u$. Thus,

$$\begin{array}{rcl} G & \to & G^0 \times G_u^u \times G^0 \\ \gamma & \to & (r(\gamma), \nu(r(\gamma))\gamma\nu^{-1}(s(\gamma)), s(\gamma)). \end{array}$$

$$\begin{tikzcd} & u & \\ s(\gamma) \arrow[ur, "\nu(s(\gamma))"] \arrow[rr, bend right, "\gamma"] & & r(\gamma) \arrow[ul, "\nu(r(\gamma))"'] \end{tikzcd}$$

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\begin{tikzcd}
 & u \arrow[dl, "\nu(s(\gamma))^{-1}"'] \arrow[loop below] & \\
s(\gamma) \arrow[rr, bend right, "\gamma"'] & & r(\gamma) \arrow[ul, "\nu(r(\gamma))"']
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More examples

Locally trivial groupoids

A transitive groupoid is *locally trivial* if there exists a family $\{U_i, \nu_i\}$ and $u \in G^0$ such that:

- $\{U_i\}$ open cover of G^0.
- $\nu_i : U_i \to G^u$ continuous, $\nu_i(v) \in G^u_v$.

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Examples

- $X \times X$, $X \times H \times X$,
- the **fundamental groupoid** of a path connected, locally connected, semi-locally simply connected space,
- the groupoid associated to a smooth transitive action of a Lie group.

Fourier-Stieltjes and Fourier algebras

Representations

Let $\mathcal{H} = (\{\mathcal{H}^u\}_{u \in G^0}, \Gamma)$ be a continuous field of separable Hilbert spaces.
$\mathcal{H}$ is a *G-Hilbert bundle* if $\forall \gamma \in G$, there exists L_γ such that:

1. $L_\gamma : \mathcal{H}^{s(\gamma)} \to H^{r(\gamma)}$ unitary isom. of Hilbert spaces.
2. $L(u) = \mathrm{Id}_{\mathcal{H}^u}$, for all $u \in G^0$.
3. $L(\delta)L(\gamma) = L(\delta\gamma)$, if $(\delta, \gamma) \in G^2$.
4. If ξ, η are continuous and bounded sections of the bundle, the map $(\xi, \eta) : G \to \mathbb{C}\ \gamma \to \langle L_\gamma \xi(s(\gamma)), \eta(r(\gamma))\rangle$ is continuous.

The Fourier algebra $B(G)$

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$$B(G) = \{(\xi, \eta) :\ \xi, \eta \in \mathcal{S}C_b(\mathcal{H}),\ \mathcal{H}\ G\text{-Hilbert bndl}\}$$

$$\varphi \in B(G), \quad \|\varphi\|_{B(G)} = \inf_{(\xi,\eta)=\varphi} \|\xi\|\|\eta\|$$

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Theorem (Paterson, 2004, Ramsay and Walter, 1997, Renault, 1997)

$B(G)$ with point-wise operations and $\|\cdot\|_{B(G)}$ is a commutative, unital Banach algebra.

Example: Left regular representation.

Haar system

A *left Haar system* is a family $\{\lambda^u\}_{u\in G^0}$ of positive regular Borel measures on G^u s. t.:

- $\text{supp}(\lambda^u) = G^u$.
- If $f \in C_c(G)$, the map $G^0 \to \mathbb{C}$, $u \to \int f\, d\lambda^u$ is continuous.
- $\forall \gamma \in G$ and $f \in C_c(G)$,

$$\int f(\gamma\gamma')\, d\lambda^{s(\gamma)}(\gamma') = \int f(\gamma')\, d\lambda^{r(\gamma)}(\gamma').$$

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$$\int f(\gamma\gamma')\, d\lambda^{s(\gamma)}(\gamma') = \int f(\gamma')\, d\lambda^{r(\gamma)}(\gamma').$$

If $\{\lambda^u\}_{u\in G^0}$ is a Haar system, we define the G-Hilbert bundle

$$L^2(G) = (\{L^2(G^u, \lambda^u)\}_{u\in G^0}, C_c(G), L),$$

where $L_\gamma(f)(\gamma') = f(\gamma^{-1}\gamma')$.

The Fourier algebra $A(G)$

A(G)

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$$\|\varphi\|_B = \inf_{(\xi,\eta)=\varphi} \|\xi\|\|\eta\| \leq \inf_{L^2\text{coeff.}} \|\xi\|\|\eta\| = \|\varphi\|_A$$

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$A(G) = \overline{A_s(G)}^{\|\cdot\|_A}$ is the **Fourier algebra** of G.

Properties of $A(G)$

If G is **proper** (i.e. $r \times s : G \to G^0 \times G^0$ is proper), then $A(G)$ is an algebra and a $B(G)$-module.

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If G is proper, compact and transitive, $A(G) = B(G)$.

Let G is locally trivial and transitive. Fix $u \in G^0$. Then,

$$A(G) = \{(\xi, \eta) : \xi, \eta \in C_0(G^0, L^2(G^u; l^2))\}.$$

Examples

If G is a l.c. group, $A(G)$ and $B(G)$ coincide with the algebras defined by Eymard.

If $G = I_n = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$, $B(I_n) = A(I_n) = M_n = \mathbb{C}^n \overset{h}{\otimes} \mathbb{C}^n$

If $G = G^0 \times \tilde{G} \times G^0$, $|G^0| = n$,

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If $G = G^0 \times \tilde{G} \times G^0$, $|G^0| = n$,

$$A(G) = \mathbb{C}^n \overset{h}{\otimes} A(\tilde{G}) \overset{h}{\otimes} \mathbb{C}^n.$$

A decomposition of the Fourier algebra.

Theorem

Let G be a locally trivial, transitive groupoid. Fix $u \in G^u$.
Suppose $\{\lambda^u\}_{u \in G^0}$ is a left Haar system such that $\lambda^u_{|_{G^u_u}} = \mu$ is the left Haar measure on G^u_u.

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Suppose $\{\lambda^u\}_{u \in G^0}$ is a left Haar system such that $\lambda^u_{|_{G^u_u}} = \mu$ is the left Haar measure on G^u_u.
Then, as Banach algebras,

$$A(G) \simeq C_0(G^0) \overset{h}{\otimes} A(G^u_u) \overset{h}{\otimes} C_0(G^0)$$

Thus, we can consider on $A(G)$ an operator space structure.
Moreover, the right hand side is a completely contractive Banach algebra, then so is $A(G)$.

Proof

$$
\begin{tikzcd}
C_0(G^0,\overline{\mathcal{H}}_r)\overset{h}{\otimes} C_0(G^0,\mathcal{H}_c) \arrow[rr, equal] \arrow[d, "\psi"'] \arrow[dr, "\pi_\psi"] & & (C_0(G^0)\overset{\vee}{\otimes}\overline{\mathcal{H}}_r)\overset{h}{\otimes}(\mathcal{H}_c\overset{\vee}{\otimes} C_0(G^0)) \arrow[rr, equal] & & C_0(G^0)\overset{h}{\otimes}(\overline{\mathcal{H}}_r\overset{\wedge}{\otimes}\mathcal{H}_c)\overset{h}{\otimes} C_0(G^0) \arrow[d, "Q"] \arrow[dl, "\pi_Q"'] \\
A(G) & \frac{\mathcal{V}}{\mathrm{Ker}(\psi)} \arrow[l, dotted, "\overline{\psi}"] & & \frac{\mathcal{V}}{\mathrm{Ker}(Q)} \arrow[ll, "\varphi"'] \arrow[r, dotted, "\overline{Q}"'] & C_0(G^0)\overset{h}{\otimes} A(\tilde{G})\overset{h}{\otimes} C_0(G^0)
\end{tikzcd}
$$

Notation

$\mathcal{H} = L^2(G^u, \lambda^u)$

$\mathcal{V} = C_0(G^0, \overline{\mathcal{H}}_r) \overset{h}{\otimes} C_0(G^0, \mathcal{H}_c).$

Thanks!