

The (continuous) Fourier algebra of a locally compact groupoid

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Groupoids

A *groupoid* is a set G together with a subset $G^2 \subseteq G \times G$, an associative product $\left\{ \begin{array}{ccc} G^2 & \rightarrow & G \\ (\delta, \gamma) & \rightarrow & \delta\gamma \end{array} \right.$ and an inverse $\left\{ \begin{array}{ccc} G & \rightarrow & G \\ \gamma & \rightarrow & \gamma^{-1} \end{array} \right.$ such that:

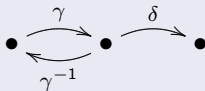
$(\gamma^{-1}, \gamma) \in G^2, \forall \gamma \in G$ and if $(\delta, \gamma) \in G^2$ then $\delta^{-1}\delta\gamma = \gamma, \delta\gamma\gamma^{-1} = \delta$.

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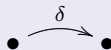


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Notation

$G^0 = \{\gamma^{-1}\gamma : \gamma \in G\}$ are the *units* of G .

The range map is $r : G \rightarrow G^0$, $r(\gamma) = \gamma\gamma^{-1}$,

The source map is $s : G \rightarrow G^0$, $s(\gamma) = \gamma^{-1}\gamma$.

If $u \in G^0$, $G^u = r^{-1}(u)$, $G_u = s^{-1}(u)$ and $G_u^u = G^u \cap G_u$ (the *isotropy group at u*).

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G is a *locally compact groupoid* if G , G^0 are l.c. Hausdorff 2nd countable spaces and the operations are continuous.

Examples

Groups

Groups are groupoids with $G^0 = \{e\}$.

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Equivalence relations

If R is an equivalence relation on a set U , it is a groupoid $G_R \subseteq U \times U$ with the following structure:

- $((w, v'), (v, u)) \in G^2 \Leftrightarrow v = v',$
 $(w, v)(v, u) = (w, u).$
- $(v, u)^{-1} = (u, v),$
- $r(v, u) = (v, v) \simeq v, s(v, u) = (u, u) \simeq u.$

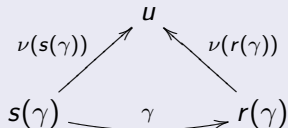
More examples

Finite case

Let G be a **finite** and **transitive** groupoid ($G_v^u \neq \emptyset$).

- Then $G(u) \cong G(v)$, $\forall u, v \in G^0$.
- Fix $u \in G^0$. Then $G \cong G^0 \times G_u^u \times G^0$:
define $\nu : G^0 \rightarrow G^u$, $\nu(v) \in G_v^u$. Thus,

$$\begin{aligned} G &\rightarrow G^0 \times G_u^u \times G^0 \\ \gamma &\rightarrow (r(\gamma), \nu(r(\gamma))\gamma\nu^{-1}(s(\gamma)), s(\gamma)). \end{aligned}$$



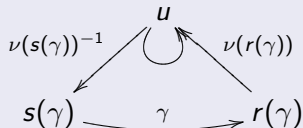
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Locally trivial groupoids

A transitive groupoid is *locally trivial* if there exists a family $\{U_i, \nu_i\}$ and $u \in G^0$ such that:

- $\{U_i\}$ open cover of G^0 .
- $\nu_i : U_i \rightarrow G^u$ continuous, $\nu_i(v) \in G_v^u$.

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Examples

- $X \times X, X \times H \times X,$
- the **fundamental groupoid** of a path connected, locally connected, semi-locally simply connected space,
- the groupoid associated to a smooth transitive action of a Lie group.

Representations

Let $\mathcal{H} = (\{\mathcal{H}^u\}_{u \in G^0}, \Gamma)$ be a continuous field of separable Hilbert spaces. $\mathcal{H}$ is a *G-Hilbert bundle* if $\forall \gamma \in G$, there exists L_γ such that:

- ① $L_\gamma : \mathcal{H}^{s(\gamma)} \rightarrow \mathcal{H}^{r(\gamma)}$ unitary isom. of Hilbert spaces.
- ② $L(u) = \text{Id}_{\mathcal{H}^u}$, for all $u \in G^0$.
- ③ $L(\delta)L(\gamma) = L(\delta\gamma)$, if $(\delta, \gamma) \in G^2$.
- ④ If ξ, η are continuous and bounded sections of the bundle, the map $(\xi, \eta) : G \rightarrow \mathbb{C} \quad \gamma \mapsto \langle L_\gamma \xi(s(\gamma)), \eta(r(\gamma)) \rangle$ is continuous.

The Fourier algebra $B(G)$

$B(G)$

$$B(G) = \{(\xi, \eta) : \xi, \eta \in \mathcal{SC}_b(\mathcal{H}), \mathcal{H} \text{ } G\text{-Hilbert bndl}\}$$

$$\varphi \in B(G), \quad \|\varphi\|_{B(G)} = \inf_{(\xi, \eta) = \varphi} \|\xi\| \|\eta\|$$

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Theorem (Paterson, 2004, Ramsay and Walter, 1997, Renault, 1997)

$B(G)$ with point-wise operations and $\|\cdot\|_{B(G)}$ is a commutative, unital Banach algebra.

Example: Left regular representation.

Haar system

A *left Haar system* is a family $\{\lambda^u\}_{u \in G^0}$ of positive regular Borel measures on G^u s. t.:

- $\text{supp}(\lambda^u) = G^u$.
- If $f \in C_c(G)$, the map $G^0 \rightarrow \mathbb{C}$, $u \rightarrow \int f d\lambda^u$ is continuous.
- $\forall \gamma \in G$ and $f \in C_c(G)$,

$$\int f(\gamma\gamma') d\lambda^{s(\gamma)}(\gamma') = \int f(\gamma') d\lambda^{r(\gamma)}(\gamma').$$

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If $\{\lambda^u\}_{u \in G^0}$ is a Haar system, we define the G -Hilbert bundle

$$L^2(G) = (\{L^2(G^u, \lambda^u)\}_{u \in G^0}, C_c(G), L),$$

where $L_\gamma(f)(\gamma') = f(\gamma^{-1}\gamma')$.

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$A(G) = \overline{A_s(G)}^{\|\cdot\|_A}$ is the **Fourier algebra** of G .

Properties of $A(G)$

If G is **proper** (i.e. $r \times s : G \rightarrow G^0 \times G^0$ is proper), then $A(G)$ is an algebra and a $B(G)$ -module.

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If G is proper, compact and transitive, $A(G) = B(G)$.

Let G is locally trivial and transitive. Fix $u \in G^0$. Then,

$$A(G) = \{(\xi, \eta) : \xi, \eta \in C_0(G^0, L^2(G^u; l^2))\}.$$

Examples

If G is a l.c. group, $A(G)$ and $B(G)$ coincide with the algebras defined by Eymard.

If $G = I_n = \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$, $B(I_n) = A(I_n) = M_n = \mathbb{C}^n \overset{h}{\otimes} \mathbb{C}^n$

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$$A(G) = \mathbb{C}^n \overset{h}{\otimes} A(\tilde{G}) \overset{h}{\otimes} \mathbb{C}^n.$$

A decomposition of the Fourier algebra.

Theorem

Let G be a locally trivial, transitive groupoid. Fix $u \in G^u$. Suppose $\{\lambda^u\}_{u \in G^0}$ is a left Haar system such that $\lambda^u|_{G_u^u} = \mu$ is the left Haar measure on G_u^u .

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Then, as Banach algebras,

$$A(G) \simeq C_0(G^0) \overset{h}{\otimes} A(G_u^u) \overset{h}{\otimes} C_0(G^0)$$

Thus, we can consider on $A(G)$ an operator space structure.

Moreover, the right hand side is a completely contractive Banach algebra, then so is $A(G)$.

Proof

$$\begin{array}{ccccc}
 C_0(G^0, \overline{\mathcal{H}}_r) \otimes^h C_0(G^0, \mathcal{H}_c) & \xlongequal{\quad} & (C_0(G^0) \overset{\vee}{\otimes} \overline{\mathcal{H}}_r) \otimes^h (\mathcal{H}_c \overset{\vee}{\otimes} C_0(G^0)) & \xlongequal{\quad} & C_0(G^0) \otimes^h (\overline{\mathcal{H}}_r \overset{\wedge}{\otimes} \mathcal{H}_c) \otimes^h C_0(G^0) \\
 \downarrow \psi & \searrow \pi_\psi & & \swarrow \pi_Q & \downarrow Q \\
 A(G) & \xleftarrow[\overline{\psi}]{} \mathcal{V} \xleftarrow[\text{Ker}(\psi)]{\varphi} \mathcal{V} \xrightarrow[\overline{Q}]{} & & & C_0(G^0) \otimes^h A(\tilde{G}) \otimes^h C_0(G^0)
 \end{array}$$

Notation

$$\mathcal{H} = L^2(G^u, \lambda^u)$$

$$\mathcal{V} = C_0(G^0, \overline{\mathcal{H}}_r) \otimes^h C_0(G^0, \mathcal{H}_c).$$

Thanks!