The projectivity of C^*-algebras and the topology of their spectra

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The Lifting Problem

Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{A}\text{-mod}$ be the category of left Banach $\mathcal{A}$-modules and continuous morphisms of left Banach $\mathcal{A}$-modules.

A module $P \in \mathcal{A}\text{-mod}$ is called **projective** if,

– for all $X,\ Y \in \mathcal{A}\text{-mod}$ and

– each epimorphism of left Banach $\mathcal{A}$-modules $\sigma : X \to Y$ for which there is a bounded linear operator $\alpha : Y \to X$ such that $\sigma \circ \alpha = \mathrm{id_Y}$,

any morphism of left Banach $\mathcal{A}$-modules $\phi : P \to Y$ can be lifted to a morphism of left Banach $\mathcal{A}$-modules $\varphi : P \to X$, that is, the following diagram

$$\begin{tikzcd} & P \arrow[dl, "\varphi"'] \arrow[d, "\phi"] \\ X \arrow[r, "\sigma"] & Y \end{tikzcd}$$

is commutative: $\sigma \circ \varphi = \phi$.

Examples

(i) $\mathcal{A} = \mathbf{C}$, then each left Banach $\mathcal{A}$-modules ($=$ a Banach space) X is projective in $\mathcal{A}\text{-mod}$.

(ii) Let $\mathcal{A}$ be a Banach algebra. Consider a **free Banach $\mathcal{A}$-module** $\mathcal{A}_+ \hat{\otimes} E$, where E is a Banach space and $\mathcal{A}_+$ is the Banach algebra obtained by adjoining an identity to $\mathcal{A}$, with multiplication:

$$a \cdot (b \otimes x) = (a \cdot b) \otimes x;\ a \in A,\ b \in A_+,\ x \in E.$$

Each $\mathcal{A}_+ \hat{\otimes} E$ is projective in $\mathcal{A}\text{-mod}$.

(iii) Each *unital* Banach algebra $\mathcal{A}$ is projective in $\mathcal{A}\text{-mod}$.

Projectivity in $\mathcal{A}$-mod

Proposition 1. (Helemskii) *Let $X \in \mathcal{A}$-mod and let us consider the canonical morphism*

$$\pi : A_+ \hat{\otimes} X \to X : a \otimes x \mapsto a \cdot x.$$

Then X is projective in $\mathcal{A}$-mod $\iff$ there is a morphism of left Banach $\mathcal{A}$-modules

$$\rho : X \to A_+ \hat{\otimes} X$$

such that $\pi \circ \rho = \mathrm{id}_X$.

We say that a Banach algebra $\mathcal{A}$ is **left projective** if $\mathcal{A}$ is projective in $\mathcal{A}$-mod.

Applications to $\mathcal{H}^2(A, X)$

Proposition 2. (Helemskii) *$\mathcal{A}$ is left projective $\iff$ the continuous Hochschild cohomology $\mathcal{H}^2(\mathcal{A}, X) = \{0\}$ for any right annihilator Banach $\mathcal{A}$-bimodule X.*

A Banach $\mathcal{A}$-bimodule X is *right annihilator* if

$$X \cdot \mathcal{A} = \{x \cdot a : a \in \mathcal{A},\ x \in X\} = \{0\}.$$

Projective C^*-algebras

For C^*-algebras $\mathcal{A}$, the relations between the separability, the existense of a strictly positive element in $\mathcal{A}$ and the left projectivity of $\mathcal{A}$ can be summarised thus:

$$\{\mathcal{A} \text{ is separable}\} \subsetneq \{\mathcal{A} \text{ has a strictly positive element }\} \subsetneq \{\mathcal{A} \text{ is left projective}\}.$$

The projectivity of such algebras was proved by Phillips $\&$ Raeburn and Lykova.

Recall (Aarnes and Kadison)

– Let a be a positive element in a C^*-algebra $\mathcal{A}$. We call a **strictly positive** if $f(a) > 0$ for all states f of $\mathcal{A}$.

– A C^*-algebra $\mathcal{A}$ contains a strictly positive element $\iff$ $\mathcal{A}$ has a countable increasing abelian approximate identity bounded by one. Such C^*-algebras are called σ-**unital**.

Projectivity of commutative C^*-algebras

For commutative C^*-algebras $\mathcal{A}$, we have the following relations

$$
\begin{array}{ccccc}
\mathcal{A} \text{ is separable} & \begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array} & \begin{array}{c}\mathcal{A} \text{ has a strictly} \\ \text{positive element}\end{array} & \begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array} & \begin{array}{c}\mathcal{A} \text{ is} \\ \text{left projective}\end{array} \\
 & & & & \\
\Updownarrow & & \begin{array}{c}\Updownarrow \\ \text{(by Aarnes and Kadison)}\end{array} & & \begin{array}{c}\Updownarrow \\ \text{(by Helemskii)}\end{array} \\
 & & & & \\
\begin{array}{c}\widehat{\mathcal{A}} \text{ is} \\ \text{metrizable and has} \\ \text{a countable base}\end{array} & \begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array} & \widehat{\mathcal{A}} \text{ is } \sigma\text{-compact} & \begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array} & \begin{array}{c}\widehat{\mathcal{A}} \text{ is} \\ \text{paracompact.}\end{array}
\end{array}
\tag{1}
$$

Here $\widehat{A}$ is the space of characters of $\mathcal{A}$ with the relative weak-* topology from the dual space of $\mathcal{A}$; $\widehat{A}$ is called the **spectrum** of $\mathcal{A}$.
Recall that $\widehat{A}$ is a Hausdorff locally compact space and $\mathcal{A} = C_0(\widehat{\mathcal{A}})$.

A topological space X is **paracompact** if each open covering of X possesses an open locally finite refinement.

Remark 1. (i) A Hausdorff locally compact space Ω may be σ-compact without having a countable base for its topology, so $\mathcal{A} = C_0(\Omega)$ may have a strictly positive element without being separable.

(ii) There are paracompact spaces which are not σ-compact.

For example, any metrizable space is paracompact, but is not always σ-compact. The simple example $\mathcal{A} = C_0(\mathbb{R})$ where $\mathbb{R}$ is endowed with the discrete topology is a left projective C^*-algebra without strictly positive elements.

Suslin condition on $\widehat{\mathcal{A}}$

The topological space Ω satisfies **the Suslin condition** if every family of non-empty open, pairwise disjoint subsets of a topological space Ω has cardinality $\leq \aleph_0$.

A topological space Ω is called a **Lindelöf space** if each open cover of Ω has a countable subcover.

Engelking's book: If a topological space Ω satisfies the Suslin condition then
Ω is paracompact $\iff$ Ω is σ-compact $\iff$ Ω is a Lindelöf space.

Lemma 1. *Let $\mathcal{A}$ be a commutative C^*-algebra contained in $\mathcal{B}(H)$, where H is a separable Hilbert space. Then the spectrum $\widehat{\mathcal{A}}$ of $\mathcal{A}$ satisfies the Suslin condition.*

$C_0(\widehat{\mathcal{A}})$ such that $\widehat{\mathcal{A}}$ satisfy the Suslin condition

Theorem 1. *Let $\mathcal{A}$ be a commutative C^*-algebra, so that $\mathcal{A} = C_0(\widehat{\mathcal{A}})$, and let $\widehat{\mathcal{A}}$ satisfy the Suslin condition. Then the following are equivalent:*

(i) *$\mathcal{A}$ is left projective;*

(ii) *the spectrum $\widehat{\mathcal{A}}$ of $\mathcal{A}$ is paracompact;*

(iii) *$\mathcal{A}$ contains a strictly positive element;*

(iv) *the spectrum $\widehat{\mathcal{A}}$ is σ-compact;*

(v) *$\widehat{\mathcal{A}}$ is a Lindelöf space;*

(vi) *$\mathcal{A}$ has a sequential approximate identity bounded by one;*

(vii) *$\mathcal{H}^2(\mathcal{A}, X) = \{0\}$ for any right annihilator Banach $\mathcal{A}$-bimodule X.*

Aarnes and Kadison, Engelking, Doran and J. Wichmann, Helemskii

Example of a nonseparable, hereditarily projective, commutative C^*-algebra $\mathcal{A} \subset \mathcal{B}(H)$, where H is a separable Hilbert space

We say a Banach algebra $\mathcal{A}$ is **hereditarily projective** if every closed left ideal of $\mathcal{A}$ is projective.

Let Ω be a Hausdorff compact topological space.

Recall

– A commutative C^*-algebra $C(\Omega)$ is separable if and only if Ω is metrizable.

– If Ω be a separable Hausdorff locally compact topological space then the C^*-algebra $C_0(\Omega)$ is contained in $\mathcal{B}(H)$ for some separable Hilbert space H.

Thus it is enough to present a Hausdorff compact topological space Ω which is separable and hereditarily paracompact, but not metrizable.

Two arrows of Alexandrov

The topological space “two arrows of Alexandrov” satisfies the above conditions.

To describe the space, let us consider two intervals $X = [0,1)$ and $X' = (0,1]$ situated one above the other. Let $\tilde{X} = X \cup X'$.

A base for the topology of $\tilde{X}$ consists of all sets of the forms

$$U = [\alpha, \beta) \cup (\alpha', \beta') \text{ and } V = (\alpha, \beta) \cup (\alpha', \beta'],$$

where $[\alpha, \beta) \subset X$, while (α', β') is the projection of $[\alpha, \beta)$ on X';
and $(\alpha', \beta'] \subset X'$, while (α, β) is the projection of $(\alpha', \beta']$ on X.

Hereditarily paracompact

A topological space X is **hereditarily paracompact** if each its subspace is paracompact.

Example 1. (i) By Stone's theorem, each metrizable topological space is paracompact, and so is hereditarily paracompact.

Recall that a unital separable commutative C^*-algebra $C(\widehat{A})$ has a metrizable spectrum $\widehat{A}$. Therefore, it is hereditarily projective.

(ii) The spectrum of a commutative von Neumann algebra is extremally disconnected, so it is not hereditarily paracompact. Therefore a commutative von Neumann algebra is not hereditarily projective. For example, l^∞ is not hereditarily projective.

Projectivity of $C_0(\widehat{A} \times \widehat{A})$

Theorem 2. *Let $\mathcal{A}$ be a $commutative$ C^*-algebra contained in $\mathcal{B}(H)$, where H is a separable Hilbert space. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is separable;*

(ii) *the C^*-tensor product*

$$\mathcal{A} \otimes_{min} A = C_0(\widehat{A} \times \widehat{A})$$

is hereditarily projective;

(iii) *the spectrum $\widehat{A}$ is metrizable and has a countable base.*

Commutative C^*-subalgebras of hereditarily projective C^*-algebras

Theorem 3. *Let $\mathcal{A}$ be a C^*-algebra, let B be a commutative C^*-subalgebra of $\mathcal{A}$ and let I be a closed ideal of B.*

Suppose that $\overline{AI}$ is projective in $\mathcal{A}$-mod.

Then I is projective in B-mod.

Theorem 4. *Every commutative C^*-subalgebra of a hereditarily projective C^*-algebra is hereditarily projective.*

Theorem 5. *No infinite-dimensional von Neumann algebra is hereditarily projective.*

Continuous fields of Banach and C^*-algebras

The basic examples of C^*-algebras defined by continuous fields are the following:

– the C^*-algebra $C_0(\Omega, \mathcal{A})$ of the constant field over a Hausdorff locally compact space Ω defined by a C^*-algebra $\mathcal{A}$;

– the direct sum of the C^*-algebras $(A_\lambda)_{\lambda\in\Lambda}$, where Λ has the discrete topology;

– there exists a canonical bijective correspondence between the liminal C^*-algebras with Hausdorff spectrum Ω and the continuous fields of non-zero elementary C^*-algebras over Ω.

Remark: Continuous fields of C^*-algebras were found useful in the characterisation of exactness and nuclearity of C^*-algebras (E. Kirchberg and S. Wassermann).

Continuous fields of Banach and C^*-algebras

Definition 1. (Dixmier) *A* **continuous field** $\mathcal{U}$ *of Banach algebras* (C^*-*algebras*) *is a triple* $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ *where*

Ω *is a locally compact Hausdorff space,*

$(A_t)_{t\in\Omega}$ *is a family of Banach algebras* (C^*-*algebras*) *and*

Θ *is an (involutive) subalgebra of* $\prod_{t\in\Omega} A_t$ *such that*

(i) *for every* $t \in \Omega$, *the set of* $x(t)$ *for* $x \in \Theta$ *is dense in* A_t*;*

(ii) *for every* $x \in \Theta$, *the function* $t \to \|x(t)\|$ *is continuous on* Ω*;*

(iii) *whenever* $x \in \prod_{t\in\Omega} A_t$ *and, for every* $t \in \Omega$ *and every* $\varepsilon > 0$, *there is an* $x' \in \Theta$ *such that* $\|x(t) - x'(t)\| \leq \varepsilon$ *throughout some neighbourhood of* t, *it follows that* $x \in \Theta$.

The elements of Θ *are called the* **continuous vector fields** *of* $\mathcal{U}$.

Continuous fields of Banach and C^*-algebras

Example 2. Let $\mathcal{A}$ be a Banach algebra (C^*-algebra),

let Ω be a locally compact Hausdorff space, and

let Θ be the (involutive) algebra of continuous mappings of Ω into $\mathcal{A}$.

For every $t \in \Omega$, put $A_t = A$.

Then $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ is a continuous field of Banach algebras (C^*-algebras) over Ω, called the **constant field** over Ω defined by A.

A field isomorphic to a constant field is said to be **trivial**.

If every point of Ω possesses a neighbourhood V such that $\mathcal{U}|_V$ is trivial, then $\mathcal{U}$ is said to be **locally trivial**.

Example 3. If Ω is discrete in Definition 1, then Axioms (i) and (iii) imply that Θ must be equal to $\prod_{t\in\Omega} A_t$.

C^*-algebras defined by continuous fields

Definition 2. *Let $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a continuous field of Banach algebras (C^*-algebras) over a locally compact Hausdorff space Ω.*

Let $\mathcal{A}$ be the set of $x \in \Theta$ such that $\|x(t)\|$ vanishes at infinity on Ω.

Then $\mathcal{A}$ with $\|x\| = \sup_{t\in\Omega} \|x(t)\|$ is a Banach algebra (C^*-algebra) which we call **the Banach algebra (C^*-algebra) defined by** *$\mathcal{U}$.*

In this talk results on the projectivity of C^*-algebras of continuous fields are joint work with my research student David Cushing.

Uniformly left projective Banach algebras A_x, $x \in \Omega$

We say that the Banach algebras A_x, $x \in \Omega$, are **uniformly left projective** if, for every $x \in \Omega$, there is a morphism of left Banach A_x-modules

$$\rho_{A_x} : A_x \to (A_x)_+ \widehat{\otimes} A_x$$

such that $\pi_{A_x} \circ \rho_{A_x} = \mathrm{id}_{A_x}$ and $\sup_{x \in \Omega} \|\rho_{A_x}\| < \infty$.

Proposition 3. *Let Ω be a locally compact Hausdorff space, let $\mathcal{U} = \{\Omega, A_t, \Theta\}$ be a locally trivial continuous field of Banach algebras, and let the Banach algebra $\mathcal{A}$ be defined by $\mathcal{U}$.*

Suppose $\mathcal{A}$ is projective in $\mathcal{A}$-mod (mod-$\mathcal{A}$, $\mathcal{A}$-mod-$\mathcal{A}$).

Then the Banach algebras A_x, $x \in \Omega$, are uniformly left (right, bi-)projective.

Let Ω be a Hausdorff locally compact space, and let $\mathcal{U} = \{\Omega, A_t, \Theta\}$ be a continuous field of Banach algebras. For any condition Γ, we say that $\mathcal{U}$ *locally satisfies condition* Γ if, for every $t \in \Omega$, there exists an open neighbourhood V of t such that $\mathcal{U}|_V$ satisfies condition Γ.

Definition 3. *We say that* $\mathcal{U}$ σ-**locally (**n-**locally) satisfies a condition** Γ *if there is an open cover* $\{U_\mu\}$, $\mu \in \mathcal{M}$, *of* Ω *such that each* $\mathcal{U}|_{U_\mu}$ *satisfies the condition* Γ *and, in addition, there is a* countable (cardinality n, respectively) *open cover* $\{V_j\}$ *of* Ω *such that* $\overline{V_j} \subset U_{\mu(j)}$ *for each* j *and some* $\mu(j) \in \mathcal{M}$.

Definition 4. *Let* Ω *be a* disjoint union *of a family of open subsets* $\{W_\mu\}$, $\mu \in \mathcal{M}$, *of* Ω. *We say that* $\mathcal{U} = \{\Omega, A_t, \Theta\}$ **is a disjoint union of** $\mathcal{U}|_{W_\mu}$, $\mu \in \mathcal{M}$.

Locally trivial continuous fields of unital Banach algebras

Theorem 6. *Let* Ω *be a Hausdorff locally compact space, let* $\mathcal{U} = \{\Omega, A_t, \Theta\}$ *be a* locally trivial *continuous field of Banach algebras such that every* $\mathcal{A}_t$ *has an identity* e_{A_t}, $t \in \Omega$, *and* $\sup_{t\in\Omega} \|e_{A_t}\| \leq C$ *for some constant* C, *and let the Banach algebra* $\mathcal{A}$ *be defined by* $\mathcal{U}$. *Then the following conditions are equivalent:*

(i) Ω *is paracompact;*

(ii) $\mathcal{A}$ *is left projective;*

(iii) $\mathcal{A}$ *is left projective and* $\mathcal{U}$ *is a disjoint union of* σ*-locally trivial continuous fields of Banach algebras;*

(iv) $\mathcal{H}^2(\mathcal{A}, X) = \{0\}$ *for any right annihilator Banach* $\mathcal{A}$*-bimodule* X.

$\mathcal{A}$ of continuous fields which is not projective

Example 4. Let Ω be $\mathbb{N}$ with the discrete topology. Consider a continuous field of Banach algebras $\mathcal{U} = \{\mathbb{N}, A_t, \prod_{t \in \mathbb{N}} A_t\}$ where A_t is the Banach algebra ℓ^2_t of t-tuples of complex numbers $x = (x_1, \ldots, x_t)$ with pointwise multiplication and the norm $\|x\| = (\sum_{i=1}^{t} |x_i|^2)^{\frac{1}{2}}$. Let $\mathcal{A}$ be defined by $\mathcal{U}$.

It is easy to see that, for each $t \in \mathbb{N}$, the algebra A_t is biprojective.

Note that the algebra A_t has the identity $e_{\ell^2_t} = (1, \ldots, 1)$ and $\|e_{\ell^2_t}\| = \sqrt{t}$. Thus $\sup_{t \in \mathbb{N}} \|e_{\ell^2_t}\| = \infty$.

One can show that the Banach algebras A_t, $t \in \mathbb{N}$, are not uniformly left projective. Therefore, $\mathcal{A}$ *is not left projective.*

Locally trivial continuous fields of C^*-algebras with strictly positive elements

Proposition 4. *Let Ω be a Hausdorff locally compact space, and let $\mathcal{A}$ be a C*-algebra which contains a strictly positive element. Suppose that Ω is paracompact. Then $C_0(\Omega, \mathcal{A})$ is left projective.*

Theorem 7. *Let Ω be a Hausdorff locally compact space with the topological dimension $\dim \Omega \leq \ell$, for some $\ell \in \mathbb{N}$, let $\mathcal{U} = \{\Omega, A_x, \Theta\}$ be a locally trivial continuous field of C^*-algebras with strictly positive elements, and let the C^*-algebra $\mathcal{A}$ be defined by $\mathcal{U}$. Then the following conditions are equivalent:*

(i) Ω *is paracompact;*

(ii) $\mathcal{A}$ *is left projective and $\mathcal{U}$ is a disjoint union of σ-locally trivial continuous fields of C^*-algebras with strictly positive elements.*

Biprojectivity of Banach algebras of continuous fields

Let $(A_\lambda)_{\lambda\in\Lambda}$ be a family of C^*-algebras. Let $\mathcal{A}$ be the set of

$$x = (x_\lambda) \in \prod_{\lambda\in\Lambda} A_\lambda$$

such that, for every $\varepsilon > 0$, $\|x_\lambda\| < \varepsilon$ except for finitely many λ. Let $\|x\| = \sup_{\lambda\in\Lambda} \|x_\lambda\|$; then $\mathcal{A}$ with $\|\cdot\|$ is a C^*-algebra and is called **the direct sum** or **the bounded product** of the C^*-algebras $(A_\lambda)_{\lambda\in\Lambda}$.

Recall Selivanov's result that any biprojective C^*-algebra is the direct sum of C^*-algebras of the type $M_n(\mathbb{C})$. Therefore one can see that biprojective C^*-algebras can be described as C^*-algebras $\mathcal{A}$ defined by a continuous field $\mathcal{U} = \{\Lambda, A_x, \prod_{x\in\Lambda} A_x\}$ where Λ has the discrete topology and the C^*-algebras A_x, $x \in \Lambda$, are of the type $M_n(\mathbb{C})$.

Theorem 8. *Let Ω be a Hausdorff locally compact space, let $\mathcal{U} = \{\Omega, A_x, \Theta\}$ be a locally trivial continuous field of Banach algebras, and let the Banach algebra $\mathcal{A}$ be defined by $\mathcal{U}$. Suppose that $\mathcal{A}$ is biprojective. Then* (i) *the Banach algebras A_x, $x \in \Omega$, are uniformly biprojective and* (ii) *Ω is discrete.*

Contractible Banach algebras of continuous fields

A Banach algebra $\mathcal{A}$ is said to be **contractible** if $\mathcal{A}_+$ is projective in the category of $\mathcal{A}$-bimodules.

A Banach algebra $\mathcal{A}$ is contractible $\iff$ $\mathcal{A}$ is biprojective and has an identity.

Theorem 9. *Let Ω be a Hausdorff locally compact space, let $\mathcal{U} = \{\Omega, A_x, \Theta\}$ be a locally trivial continuous field of Banach algebras, and let the Banach algebra $\mathcal{A}$ be defined by $\mathcal{U}$. Then the following conditions are equivalent:*

(i) $\mathcal{A}$ *is contractible;*

(ii) Ω *is finite and discrete, and the Banach algebras A_x, $x \in \Omega$, are contractible;*

(iii) $\mathcal{H}^n(\mathcal{A}, X) = \{0\}$ *for any Banach $\mathcal{A}$-bimodule X and all $n \geq 1$.*

Examples of biprojective Banach algebras defined by continuous fields

Example 5. Let Ω be a topological space with the discrete topology.

For every $t \in \Omega$, let E_t be an arbitrary Banach space of dimension $\dim E_t > 1$.

Take a continuous linear functional $f_t \in E_t^*$, $\|f_t\| = 1$ and define on E_t the structure of a Banach algebra $A_{f_t}(E_t)$ with multiplication given by $ab = f_t(a)b$, $a, b \in A_{f_t}(E_t)$.

Consider the continuous field of Banach algebras $\mathcal{U} = \{\Omega, A_t, \prod_{t \in \Omega} A_t\}$ where A_t is the Banach algebra $A_{f_t}(E_t)$ with $f_t \in E_t^*$, $\|f_t\| = 1$. Then the Banach algebra $\mathcal{A}$ defined by $\mathcal{U}$ is biprojective.

Since $\mathcal{A}$ is biprojective, $\mathcal{H}^n(\mathcal{A}, X) = \{0\}$ for all Banach $\mathcal{A}$-bimodule X and all $n \geq 3$ (Helemskii).

Since $\mathcal{A}$ is biprojective and has a left bounded approximate identity, $\mathcal{H}^n(\mathcal{A}, X^*) = \{0\}$ for all dual Banach $\mathcal{A}$-bimodule X^* and all $n \geq 2$ (Selivanov). $\square$

Example 6. Let Ω be a topological space with the discrete topology.

For every $t \in \Omega$, let $(E_t, F_t, \langle \cdot, \cdot \rangle_t)$ be a pair of Banach spaces with a non-degenerate continuous bilinear form $\langle x, y \rangle_t$, $x \in E_t$, $y \in F_t$, with $\| \langle \cdot, \cdot \rangle_t \| \leq 1$.

The *tensor algebra* $E_t \widehat{\otimes} F_t$ *generated by the duality* $\langle \cdot, \cdot \rangle_t$ can be constracted on the Banach space $E_t \widehat{\otimes} F_t$ where the multiplication is defined by the formula

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = \langle x_2, y_1 \rangle_t x_1 \otimes y_2, \quad x_i \in E_t, y_i \in F_t.$$

Consider the continuous field of Banach algebras $\mathcal{U} = \{\Omega, A_t, \prod_{t \in \Omega} A_t\}$ where A_t is the Banach algebra $E_t \widehat{\otimes} F_t$ with $\| \langle \cdot, \cdot \rangle_t \| \leq 1$. Then the Banach algebra $\mathcal{A}$ defined by $\mathcal{U}$ is biprojective.

Since $\mathcal{A}$ is biprojective, $\mathcal{H}^n(\mathcal{A}, X) = \{0\}$ for all Banach $\mathcal{A}$-bimodule X and all $n \geq 3$. $\square$

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Thank you