Generalized Weak Peripheral Multiplicativity in Algebras of Lipschitz Functions

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Interplay Between Structures

Let V and W be normed vector spaces over $\mathbb{R}$.

Suppose that the metric structure of V is the same as the metric structure of W. What can we say about the linear structures of V and W?

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Suppose that the metric structure of V is the same as the metric structure of W. What can we say about the linear structures of V and W?

Theorem (Mazur and Ulam '32)

Let $T\colon V \to W$ be a surjective map between normed vector spaces over $\mathbb{R}$ such that $T(0) = 0$ and

$$\|T(x) - T(y)\|_W = \|x - y\|_V$$

for all $x, y \in V$. Then T is linear.

This is a **preserver problem**, as it involves analyzing a mapping between two spaces such that a property is invariant under the mapping.

Preserver Problems in $C(X)$

Let X be a compact Hausdorff space and let $C(X)$ be the collection of all complex-valued continuous functions on X.

Given an $f \in C(X)$, we denote the **spectrum** of f by

$$\sigma(f) = \{\lambda \in \mathbb{C} \colon f - \lambda \notin C(X)^{-1}\}.$$

Suppose we had a surjective mapping $T \colon C(X) \to C(X)$ that satisfies

$$\sigma(T(f)T(g)) = \sigma(fg)$$

for all $f, g \in C(X)$.

Will the mapping T respect the algebraic and topological structures of $C(X)$?

Spectral Preserver Problem

Theorem (Molnar '01)

Let X be a compact Hausdorff space and let $T\colon C(X) \to C(X)$ be a surjective map such that

$$\sigma(T(f)T(g)) = \sigma(fg)$$

for all $f, g \in C(X)$. Then there exists a homeomorphism $\psi\colon X \to X$ such that

$$T(f)(x) = T(1)(x)f(\psi(x))$$

for all $f \in C(X)$ and all $x \in X$.

This is an example of a **spectral preserver problem**, as it involves a mapping that preserves a certain spectral property. There has been much work done on spectral preserver problems [Hatori et al., '11].

Uniform Algebras

We wish to extend, and generalize, the result of Molnar to **uniform algebras**.

Given an $f \in \mathcal{A}$, the **peripheral spectrum** is the set

$$\sigma_\pi(f) = \{z \in \sigma(f) \colon |z| = \|f\|_\infty\},$$

where $\|\cdot\|_\infty$ denotes the uniform norm.

We will investigate surjective mappings $T \colon \mathcal{A} \to \mathcal{B}$ between uniform algebras that satisfy

$$\sigma_\pi(T(f)T(g)) = \sigma_\pi(fg)$$

for all $f, g \in \mathcal{A}$.

Tools in Uniform Algebras

The **maximizing set** of $f \in \mathcal{A}$ is the set $M(f) = \{x \in X : |f(x)| = \|f\|_\infty\}$.

A point $x \in X$ is a **weak peak point** of $\mathcal{A}$ if $\{x\} = \bigcap_{h \in \mathcal{F}} M(h)$, for some collection $\mathcal{F} \subset \mathcal{A}$. The collection of weak peak points of $\mathcal{A}$ is denoted by $\delta\mathcal{A}$.

A key result is that we can multiplicatively isolate the values of functions on $\delta\mathcal{A}$.

Lemma (Bishop '59)

Let $\mathcal{A}$ be a uniform algebra; let $x \in \delta\mathcal{A}$; and let $f \in \mathcal{A}$ be such that $f(x) \neq 0$. Then there exists an $h \in \mathcal{A}$ such that $\sigma_\pi(h) = \{1\}$, $h(x) = 1$, and $\sigma_\pi(fh) = \{f(x)\}$.

Peripheral Multiplicativity

Theorem (Luttman and Tonev '07)

Let $T\colon \mathcal{A} \to \mathcal{B}$ be a surjective mapping between uniform algebras such that

$$\sigma_\pi(T(f)T(g)) = \sigma_\pi(fg)$$

for all $f, g \in \mathcal{A}$. Then there exists a homeomorphism $\psi\colon \delta\mathcal{B} \to \delta\mathcal{A}$ such that

$$T(f)(y) = T(1)(y)f(\psi(y))$$

for all $f \in \mathcal{A}$ and all $y \in \delta\mathcal{B}$.

Can we weaken $\sigma_\pi(T(f)T(g)) = \sigma_\pi(fg)$?

Weak Peripheral Multiplicativity

Theorem (L. and Luttman '11)

Let $T\colon \mathcal{A} \to \mathcal{B}$ be a surjective mapping between uniform algebras on **first-countable**, *compact Hausdorff spaces X and Y, respectively, such that*

$$\sigma_\pi(T(f)T(g)) \cap \sigma_\pi(fg) \neq \varnothing$$

holds for all $f, g \in \mathcal{A}$. Then there exists a homeomorphism $\psi\colon \delta\mathcal{B} \to \delta\mathcal{A}$ such that

$$T(f)(y) = T(1)(y) f(\psi(y))$$

for all $f \in \mathcal{A}$ and all $y \in \delta\mathcal{B}$.

It is unknown if the assumption of first-countability can be dropped.

WPM Multi-Map

Recently, there has been work done in analyze pairs of mappings that jointly satisfy spectral conditions [Hatori et al., '10].

Theorem (L. and Luttman '11)

Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras on **first-countable**, *compact Hausdorff spaces X and Y, respectively, and let $T_1, T_2 \colon \mathcal{A} \to \mathcal{B}$ be surjective mappings such that*

$$\sigma_\pi(T_1(f)T_2(g)) \cap \sigma_\pi(fg) \neq \varnothing$$

for all $f, g \in \mathcal{A}$. Then $T_1(1)T_2(f) = T_1(f)T_2(1)$ for all $f \in \mathcal{A}$.

The map $\Phi(f) = T_1(f)T_2(1)$ satisfies $\sigma_\pi(\Phi(f)\Phi(g)) \cap \sigma_\pi(fg) \neq \varnothing$ for all $f, g \in \mathcal{A}$.

Lipschitz Algebras

Let (X, d_X) be a compact metric space with a distinguished base point e_X.

The collection of all complex-valued **Lipschitz functions** that vanish at e_X is the set

$$\text{Lip}_0(X) = \left\{ f \in C(X) \colon \sup_{x,y \in X, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} = L_{d_X}(f) < \infty, f(e_X) = 0 \right\}$$

Note that $\text{Lip}_0(X)$ is a (weak) Banach algebra with respect to the norm $L_{d_X}(\cdot)$, but $\text{Lip}_0(X)$ is not a uniform algebra.

Can we extend the results for uniform algebras to $\text{Lip}_0(X)$?

Tools in Lipschitz Algebras

First, $\text{Lip}_0(X)$ need not have a multiplicative identity, so $\sigma(f)$ may not exist.

However, the **peripheral range** of f always exists and is denoted by

$$\text{Ran}_\pi(f) = \{\lambda \in \text{Ran}(f) \colon |\lambda| = \|f\|_\infty\}.$$

For each $x \in X \setminus \{e_X\}$, there exists an $h \in \text{Lip}_0(X)$ such that $M(h) = \{x\}$.

A stronger version of Bishop's Lemma holds for $\text{Lip}_0(X)$.

Lemma (Jimenez-Vargas, Luttman, and Villegas-Vallecillos, '10)

Let $x \in X \setminus \{e_X\}$ and let $f \in \mathcal{A}$ be such that $f(x) \neq 0$. Then there exists an $h \in \text{Lip}_0(X)$ such that $\text{Ran}_\pi(f) = \{1\}$ and $\{x\} = M(h) = M(fh)$.

Lipschitz WPM

Our goal is to characterize the form of surjective mappings $T_1, T_2 \colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ and $S_1, S_2 \colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(X)$ that satisfy

$$\mathrm{Ran}_\pi(T_1(f)T_2(g)) \cap \mathrm{Ran}_\pi(S_1(f)S_2(g)) \neq \varnothing. \quad (1)$$

Based on the previous results, we suspect that T_1 and T_2 are weighted composition operators. However, there are a few differences.

- $\mathrm{Lip}_0(X)$ does not contain the constant function 1.
- Equation (1) does not have f and g unaltered. S_1 and S_2 may have some impact on the form of T_1 and T_2.
- The pre-composition function $\psi \colon Y \to X$ is between metric spaces and is being composed with Lipschitz functions. Thus, in addition to being a homeomorphism, we wish ψ and ψ^{-1} to be Lipschitz.

Lipschitz Multi-Map Theorem

Theorem (Jiménez-Vargas, L., Luttman, Villegas-Vallecillos '11)

Let (X, d_X) *and* (Y, d_Y) *be pointed compact metric spaces and let* $T_1, T_2\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ *and* $S_1, S_2\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(X)$ *be surjective mappings that satisfy*

$$\mathrm{Ran}_\pi(T_1(f)T_2(g)) \cap \mathrm{Ran}_\pi(S_1(f)S_2(g)) \neq \varnothing$$

for all $f, g \in \mathrm{Lip}_0(X)$*. Then there exist mappings* $\varphi_1, \varphi_2\colon Y \to \mathbb{C}$ *with* $\varphi_1(y)\varphi_2(y) = 1$ *for all* $y \in Y$ *and a Lipschitz homeomorphism* $\psi\colon Y \to X$ *such that* $\psi(e_Y) = e_X$ *and*

$$T_j(f)(y) = \varphi_j(y)S_j(f)(\psi(y))$$

holds for all $f \in \mathrm{Lip}_0(X)$*, all* $y \in Y$*, and* $j = 1, 2$*.*

Construction of $\psi, \varphi_1, \varphi_2$

Given $x \in X \setminus \{e_X\}$, define

$$F_x(\mathrm{Lip}_0(X)) = \{f \in \mathrm{Lip}_0(X) \colon \|f\|_\infty = |f(x)| = 1\}, \text{ and}$$
$$\mathcal{P}_x(\mathrm{Lip}_0(X)) = \{f \in \mathrm{Lip}_0(X) \colon \mathrm{Ran}_\pi(f) = \{1\},\ x \in M(f)\}.$$

For each $x \in X \setminus \{e_X\}$, the set

$$A_x = \bigcap_{h \in \mathcal{A}_1, k \in \mathcal{A}_2} M(T_1(h)T_2(k))$$

is non-empty, where $\mathcal{A}_j = S_j^{-1}[F_x(\mathrm{Lip}_0(X))]$ for $j = 1, 2$.

Given $x \in X \setminus \{e_X\}$ and $y \in A_x$, we have that $S_1(f)S_2(g) \in F_x(\mathrm{Lip}_0(X))$ if and only if $T_1(f)T_2(g) \in F_y(\mathrm{Lip}_0(Y))$. This implies that A_x is a singleton.

Construction of $\psi, \varphi_1, \varphi_2$, cont.

Define $\tau \colon X \to Y$ by $\tau(e_X) = e_Y$ and, for $x \neq e_X$,

$$\{\tau(x)\} = A_x.$$

The mapping τ is a homeomorphism and we set $\psi = \tau^{-1}$.

For $x \in X \setminus \{e_X\}$, we have that $T_1(h)(\tau(x))\, T_2(k)(\tau(x)) = 1$ for all pairs (h, k) satisfying $h \in S_1^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$ and $k \in S_2^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$.

Define $\varphi_j \colon Y \to \mathbb{C}$ by $\varphi_j(e_Y) = 1$ and, for $y \neq e_Y$,

$$\varphi_j(y) = T_i(h)(y),$$

where $h \in S_j^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$, $\tau(x) = y$, and $j = 1, 2$. Note that $\varphi_1(y)\varphi_2(y) = 1$ for all $y \in Y$.

Construction of $\psi, \varphi_1, \varphi_2$, cont.

For each $f \in \text{Lip}_0(X)$, we have that $T_j(f)(y) = \varphi_j(y) S_j(f)(\psi(y))$ for all $f \in \text{Lip}_0(X)$, for all $y \in Y$, and $j = 1, 2$.

As a consequence of the following lemma, τ and ψ are Lipschitz.

Lemma (Jiménez-Vargas, L., Luttman, Villegas-Vallecillos '11)

Let (X, d_X) and (Y, d_Y) be pointed compact metric spaces and let $\psi\colon Y \to X$ be a continuous mapping. If ψ is not Lipschitz, then there exist sequences $\{y_n\}$ and $\{z_n\}$ in Y converging to a point $y \in Y$ such that $y_n \neq z_n$ and

$$n < \frac{d_X(\psi(y_n), \psi(z_n))}{d_Y(y_n, z_n)}$$

for all $n \in \mathbb{N}$ and a function $f \in \text{Lip}_0(X)$ such that $f(\psi(y_n)) = d_X(\psi(y_n), \psi(z_n))$ and $f(\psi(z_n)) = 0$ for all $n \in \mathbb{N}$.

Future Work and Acknowledgments

Future Work:

The key result to the work on peripheral multiplicativity, and its generalizations, is Bishop's Lemma.

Currently, we are investigating subalgebras of $C(X)$ where Bishop's Lemma is true.

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