A very proper Heisenberg–Lie Banach $*$-algebra

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Background

Let $q_1, q_2 \in \mathbb{C} \setminus \{0\}$. Three elements b_1, b_2 and b_3 of a (complex) algebra satisfy the (q_1, q_2)*-deformed Heisenberg–Lie commutation relations* if

$$b_1 b_2 - q_1 b_2 b_1 = b_3, \qquad q_2 b_1 b_3 - b_3 b_1 = 0 \qquad \text{and} \qquad b_2 b_3 - q_2 b_3 b_2 = 0.$$

Classical (undeformed) case: $q_1 = q_2 = 1$. Coloured case: $q_1 = q_2 = -1$.

Question (Sigurdsson–Silvestrov): Can these commutation relations be realized in a Banach algebra?

Answer (L–Silvestrov, *Math. Proc. Royal Irish Acad.* (2009)): Yes, a universal normalized solution exists.

Theorem. For $q_1, q_2 \in \mathbb{C} \setminus \{0\}$, there is a unital Banach algebra $\mathscr{B}_{q_1,q_2}$ that contains elements b_1, b_2 and b_3 which satisfy the (q_1, q_2)-deformed Heisenberg–Lie commutation relations, and

- $\|b_1\| = \|b_2\| = 1$ and $\|b_3\| = 1 + |q_1|$;
- whenever a unital Banach algebra $\mathscr{A}$ contains elements a_1, a_2 and a_3 which satisfy the (q_1, q_2)-deformed Heisenberg–Lie commutation relations and $\|a_j\| \leqslant 1$ for $j = 1, 2$, there is a unique bounded unital algebra homomorphism $\varphi\colon \mathscr{B}_{q_1,q_2} \to \mathscr{A}$ such that $\varphi(b_j) = a_j$ for $j = 1, 2, 3$.

A $*$-algebraic version

Let $q_1, q_2 \in \mathbb{R} \setminus \{0\}$. A pair of elements c_1 and c_2 of a $*$-algebra satisfy the *$*$-algebraic (q_1, q_2)-deformed Heisenberg–Lie commutation relations* if

$$c_1 c_1^* - q_1 c_1^* c_1 = c_2 \qquad \text{and} \qquad q_2 c_1 c_2 - c_2 c_1 = 0.$$

Note: In this case $b_1 = c_1$, $b_2 = c_1^*$ and $b_3 = c_2$ satisfy the (q_1, q_2)-deformed Heisenberg–Lie commutation relations.

Theorem (L–Silvestrov). For $q_1, q_2 \in \mathbb{R} \setminus \{0\}$, there is a unital Banach $*$-algebra $\mathscr{C}_{q_1,q_2}$ that contains a pair of elements c_1 and c_2 which satisfy the $*$-algebraic (q_1, q_2)-deformed Heisenberg–Lie commutation relations, and

- $\|c_1\| = 1$ and $\|c_2\| = 1 + |q_1|$;
- whenever a unital Banach $*$-algebra $\mathscr{A}$ contains a pair of elements a_1 and a_2 which satisfy the $*$-algebraic (q_1, q_2)-deformed Heisenberg–Lie commutation relations and $\|a_1\| \leqslant 1$, there is a unique bounded unital $*$-homomorphism $\varphi\colon \mathscr{C}_{q_1,q_2} \to \mathscr{A}$ such that $\varphi(c_j) = a_j$ for $j = 1, 2$.

Note: A Banach $*$-algebra has an *isometric* involution.

How about C^*-algebras?

Theorem (Generalized Gelfand–Naimark–Segal construction). A unital Banach $*$-algebra $\mathscr{A}$ has a faithful $*$-representation on a Hilbert space if and only if $\mathscr{A}$ is $*$-*semisimple* in the sense that $*\text{-rad}\,\mathscr{A} = \{0\}$, where

$$*\text{-rad}\,\mathscr{A} = \bigcap\{\ker\lambda : \lambda\colon \mathscr{A} \to \mathbb{C} \text{ is linear and positive}\}.$$

Recall: λ is *positive* $\iff$ $\langle a^*a, \lambda\rangle \geqslant 0$ for each $a \in \mathscr{A}$.

Question: Are any of the Banach $*$-algebras $\mathscr{C}_{q_1,q_2}$ $*$-semisimple?

Theorem (L–Silvestrov).

- A pair of elements c_1 and c_2 of a $*$-semisimple Banach $*$-algebra satisfy the $*$-algebraic $(1,1)$-deformed Heisenberg–Lie commutation relations if and only if c_1 is normal (in the sense that $c_1 c_1^* = c_1^* c_1$) and $c_2 = 0$.
  Hence $\mathscr{C}_{1,1}$ is not $*$-semisimple.
- Suppose that one of the following five conditions holds:
  - $q_1 < 0$ and $q_2 < 0$;
  - $0 < q_1 < 1$, and either $0 < q_2 < 1$ or $-\dfrac{1-q_1}{1+q_1} < q_2 < 0$;
  - $0 < q_2 < 1$ and $-\dfrac{1-q_2}{1+q_2} < q_1 < 0$;
  - $q_1 > 1$, and either $q_2 > 1$ or $q_2 < -\dfrac{q_1+1}{q_1-1}$ (< -1);
  - $q_2 > 1$ and $q_1 < -\dfrac{q_2+1}{q_2-1}$ (< -1).

  Then a pair of elements c_1 and c_2 of a $*$-semisimple Banach $*$-algebra satisfy the $*$-algebraic (q_1, q_2)-deformed Heisenberg–Lie commutation relations if and only if $c_1 = c_2 = 0$.
  Hence $\mathscr{C}_{q_1,q_2}$ is not $*$-semisimple.
- $\mathscr{C}_{q,-q^{-1}}$ is not $*$-semisimple for any $q \in \mathbb{R} \setminus \{0, \pm 1\}$.

Main open case: $q_1 = -q_2 = \pm 1$. **Fact.** $\mathscr{C}_{1,-1} = \mathscr{C}_{-1,1}$.

Theorem (L, *Positivity* (to appear)). $\mathscr{C}_{-1,1}$ is *very proper* in the sense that

$$\forall m \in \mathbb{N}\ \forall a_1, \ldots, a_m \in \mathscr{C}_{-1,1} \colon \ \sum_{i=1}^{m} a_i^* a_i = 0 \ \Rightarrow \ a_1 = a_2 = \cdots = a_m = 0.$$

Motivation. A very proper Banach $*$-algebra is "almost" $*$-semisimple:

Theorem (Kelley–Vaught). The $*$-radical of a unital Banach $*$-algebra $\mathscr{A}$ is given by

$$*\text{-rad}\, \mathscr{A} = \left\{ a \in \mathscr{A} : -a^* a \in \overline{\mathscr{A}^+} \right\},$$

where $\overline{\mathscr{A}^+}$ denotes the norm-closure of the positive cone

$$\mathscr{A}^+ = \left\{ \sum_{i=1}^{m} a_i^* a_i : m \in \mathbb{N},\ a_1, \ldots, a_m \in \mathscr{A} \right\}.$$

Note: $\mathscr{A}$ is very proper if and only if $\{a \in \mathscr{A} : -a^* a \in \mathscr{A}^+\} = \{0\}$.

Further motivation

Theorem (Palmer). A very proper $*$-algebra $\mathscr{A}$ is *ordered* in the sense that

$$\mathscr{A}^+ \cap (-\mathscr{A}^+) = \{0\}.$$

Hence

$$a \leqslant b \iff b - a \in \mathscr{A}^+$$

defines a partial order on the set of self-adjoint elements of $\mathscr{A}$.

Theorem (L–Silvestrov). For $q_1, q_2 < 0$, a pair of elements c_1 and c_2 of a very proper $*$-algebra satisfy the $*$-algebraic (q_1, q_2)-deformed Heisenberg–Lie commutation relations if and only if $c_1 = c_2 = 0$.

Hence $\mathscr{C}_{q_1,q_2}$ is not very proper for $q_1, q_2 < 0$.

Recall: The $*$-algebraic (q_1, q_2)-deformed Heisenberg–Lie commutation relations are

$$c_1 c_1^* - q_1 c_1^* c_1 = c_2 \qquad \text{and} \qquad q_2 c_1 c_2 - c_2 c_1 = 0.$$

Letting $c = c_1$ and suppressing c_2, we obtain

$$q_2 c^2 c^* - (1 + q_1 q_2) c c^* c + q_1 c^* c^2 = 0.$$

For $q_1 = -q_2 = \pm 1$: $\qquad c^2 c^* = c^* c^2.$

Let $\mathbb{S}_2$ be the free unital semigroup on two generators s and s^*. (Elements are "words" in s and s^*, together with the neutral element e; the product is concatenation.) This notation indicates that the map which interchanges the generators ($s \leftrightarrow s^*$) induces a semigroup involution on $\mathbb{S}_2$ (anti-multiplicative and of period two).
Hence it induces an isometric involution on the semigroup Banach algebra

$$\ell_1(\mathbb{S}_2) = \left\{ f : \mathbb{S}_2 \to \mathbb{C} : \sum_{w \in \mathbb{S}_2} |f(w)| < \infty \right\}.$$

Notation: $f = \sum_{w \in \mathbb{S}_2} f(w)\delta_w$.
Convolution product: $\delta_v \delta_w = \delta_{vw}$; involution: $\delta_w^* = \delta_{w^*}$ for $v, w \in \mathbb{S}_2$.

The definition of $\mathscr{C}_{-1,1}$ (continued)

Let $\mathscr{J}$ be the closed $*$-ideal in $\ell_1(\mathbb{S}_2)$ generated by $g = \delta_{s^2 s^*} - \delta_{s^* s^2}$. Then

$$\mathscr{C}_{-1,1} = \ell_1(\mathbb{S}_2)/\mathscr{J}.$$

Indeed, $c = \pi(\delta_s)$ satisfies $c^2 c^* = c^* c^2$, where $\pi\colon \ell_1(\mathbb{S}_2) \to \mathscr{C}_{-1,1}$ is the quotient homomorphism, so $c_1 = c$ and $c_2 = cc^* + c^* c$ satisfy the $*$-algebraic $(-1,1)$-deformed Heisenberg–Lie commutation relations.

The universality of this solution follows from the First Isomorphism Theorem and the following universal property of $\ell_1(\mathbb{S}_2)$:

> *Given an element a in the unit ball of a unital Banach $*$-algebra $\mathscr{A}$, there is a unique bounded unital $*$-homomorphism $\psi\colon \ell_1(\mathbb{S}_2) \to \mathscr{A}$ such that $\psi(\delta_s) = a$.*

Outline of the proof that $\mathscr{C}_{-1,1}$ is very proper

Let $m \in \mathbb{N}$, and suppose that $a_1, \ldots, a_m$ are non-zero elements of $\mathscr{C} := \mathscr{C}_{-1,1}$.

Aim: Prove that $\sum_{i=1}^m a_i^* a_i \neq 0$.

Strategy: Find a linear map φ with domain $\mathscr{C}$ such that $\varphi\big(\sum_{i=1}^m a_i^* a_i\big) \neq 0$.

A reduction. Let $W = \bigcup_{n=0}^\infty (X_n \cup Y_n \cup Z_n)$, where

$$X_n = \big\{s^j (s^*)^{n-j} : j \in \{0, 1, \ldots, n\}\big\},$$
$$Y_n = \big\{s^j (s^*)^{2k+1} (ss^*)^\ell : j, k \in \mathbb{N}_0,\ \ell \in \mathbb{N},\ j + 2k + 2\ell + 1 = n\big\},$$
$$Z_n = \big\{s^j (s^*)^{2k+1} (ss^*)^\ell s : j, k, \ell \in \mathbb{N}_0,\ j + 2k + 2\ell + 2 = n\big\},$$

(Note: $Y_n = \emptyset$ for $n \leqslant 2$; and $Z_n = \emptyset$ for $n \leqslant 1$.)

Lemma. For each $a \in \mathscr{C}$, there is $f \in \ell_1(\mathbb{S}_2)$ such that $\pi(f) = a$ and $f(w) = 0$ for each $w \in \mathbb{S}_2 \setminus W$.

Reason: $c^2 c^* = c^* c^2$.

Application: Write $a_i = \pi(f_i)$ for $i \in \{1, \ldots, m\}$, where $f_i(w) = 0$ for each $w \in \mathbb{S}_2 \setminus W$.

Outline proof: The operators $\widetilde{\rho}_{j,k}$

For $j, k \in \mathbb{N}_0$, let $V_{j,k} \subset \mathbb{S}_2$ be the set of words in j letters s and k letters s^*.

Examples. $V_{0,0} = \{e\}$, $V_{1,0} = \{s\}$, $V_{0,1} = \{s^*\}$, $V_{1,1} = \{ss^*, s^*s\}$ and $V_{2,1} = \{s^2s^*, ss^*s, s^*s^2\}$.

Then

$$\rho_{j,k} \colon \ f \mapsto \sum_{v \in V_{j,k}} f(v)\delta_v, \quad \ell_1(\mathbb{S}_2) \to \ell_1(\mathbb{S}_2),$$

is a projection of norm 1. The First Isomorphism Theorem: There is a unique bounded operator $\widetilde{\rho}_{j,k}$ on $\mathscr{C}$ such that

$$\begin{array}{ccc} \ell_1(\mathbb{S}_2) & \xrightarrow{\ \rho_{j,k}\ } & \ell_1(\mathbb{S}_2) \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ \mathscr{C} & \overset{\widetilde{\rho}_{j,k}}{\dashrightarrow} & \mathscr{C}. \end{array}$$

Outline proof: Finding a "witness word"

Let

$$N = \min\big\{n \in \mathbb{N}_0 : \widetilde{\rho}_{j,n-j}(a_i) \neq 0 \text{ for some } i \in \{1, \ldots, m\} \text{ and } j \in \{0, \ldots, n\}\big\}.$$

Recall: $a_i = \pi(f_i)$. We may suppose that

$$f_1, \ldots, f_m \in \ker \rho_{j,n-j} \quad \text{for} \quad n \in \{0, \ldots, N-1\} \text{ and } j \in \{0, \ldots, n\}.$$

Then

$$\widetilde{\rho}_{N,N}\Big(\sum_{i=1}^{m} a_i^* a_i\Big) = \pi\Big(\sum_{i=1}^{m}\sum_{j=0}^{N}(\rho_{j,N-j}f_i)^*(\rho_{j,N-j}f_i)\Big).$$

Recall: $\rho_{j,N-j}f_i$ is supported on words belonging to W in j letters s and $N-j$ letters s^*. For N and j even (say), these words are

$$s^j(s^*)^{N-j}, \quad s^{j-2\ell+1}(s^*)^{N-j-2\ell+1}(ss^*)^{2\ell-1} \text{ and } s^{j-2\ell}(s^*)^{N-j-2\ell+1}(ss^*)^{2\ell-1}s,$$

where $\ell \in \{1, \ldots, L\}$ and $L = \frac{1}{2}\min\{j, N-j\}$.

Key point: In the latter two cases, the "tail" $(ss^*)^{2\ell-1}$ cannot "untwist" itself in

$$(\rho_{j,N-j}f_i)^*(\rho_{j,N-j}f_i).$$

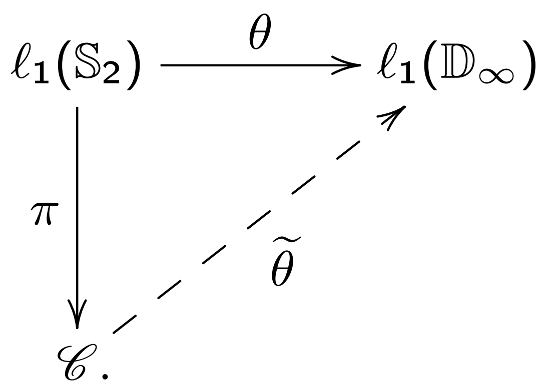
Outline proof: Formalizing the "witness word"

Let $\mathbb{D}_\infty$ be the *infinite dihedral group;* that is, $\mathbb{D}_\infty$ is the group generated by two elements r and r^* satisfying $r^2 = (r^*)^2 = e$. The map $r \leftrightarrow r^*$ induces a semigroup involution on $\mathbb{D}_\infty$. Hence $\ell_1(\mathbb{D}_\infty)$ is a Banach $*$-algebra.

Warning: This is *not* the usual involution on a group algebra ($\delta_g \mapsto \delta_{g^{-1}}$).

The universal property of $\ell_1(\mathbb{S}_2)$: There is a unique bounded unital $*$-homomorphism $\theta\colon \ell_1(\mathbb{S}_2) \to \ell_1(\mathbb{D}_\infty)$ such that $\theta(\delta_s) = \delta_r$.

The First Isomorphism Theorem: There is a unique bounded unital $*$-homomorphism $\widetilde{\theta}$ such that



Outline proof: Joining the pieces together

Suppose that N is even and that

$$f_i(s^j(s^*)^{2k+1}(ss^*)^\ell) \neq 0$$

for some $i \in \{1, \ldots, m\}$, $j, k \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$ with $N = j + 2k + 1 + 2\ell$.

Take ℓ maximal. Then

$$\widetilde{\theta}\,\widetilde{\rho}_{N,N}\left(\sum_{i=1}^{m} a_i^* a_i\right)\left((rr^*)^{2(\ell+1)}\right)$$

$$= \sum_{i=1}^{m}\sum_{j=0}^{N}(\theta\rho_{j,N-j}f_i)^*(\theta\rho_{j,N-j}f_i)\left((rr^*)^{2(\ell+1)}\right)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{\frac{N}{2}-\ell}\left|f_i\left(s^{2j-1}(s^*)^{N-2j-2\ell+1}(ss^*)^\ell\right)\right|^2 > 0,$$

so $\sum_{i=1}^{m} a_i^* a_i \neq 0$. $\square$