Automatic continuity of group representations and homomorphisms

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Definitions

G — a locally compact group
H — a Hilbert space
$\mathcal{U}(H)$ — group of unitary operators on H,
a topological group in the weak operator topology
$\pi : G \to \mathcal{U}(H)$ — a *unitary* representation
$f(t) = \langle \pi(t)x, y \rangle$ — coefficient of π

π is continuous if $\pi^{-1}(V)$ is open
for every open set $V \subset \mathcal{U}(H)$
$\Leftrightarrow$ every coefficient is continuous
π is measurable if $\pi^{-1}(V)$ is Haar measurable
for every open set $V \subset \mathcal{U}(H)$
π is weakly measurable if every coefficient is measurable

Example

Weakly measurable but discontinuous :

$G = \mathbb{T}$ — the unit circle (or any non-discrete group)
$\mathbb{T}_d$ — the circle with discrete topology
$H = \ell_2(\mathbb{T}_d)$ — 2-summable sequences,
every $x(t)$ is 0 except for t in a countable set.

G acts by translations on H, discontinuously ;
every coefficient is 0 except for a countable set :
$\langle \pi(t)x, y \rangle = \sum x(s_n - t)y(s_n)$,

$$s_n \in \operatorname{supp} y, \quad t \in s_n - \operatorname{supp} x \subset \operatorname{supp} y - \operatorname{supp} x.$$

so this representation is weakly measurable.

Theorem on continuity of representations

Theorem 1

Every measurable unitary representation is continuous.

For separable Hilbert spaces, this is a classical result.
In this case it suffices already that π is weakly measurable.

In general, as known, a weakly measurable representation can be decomposed as $\pi = \pi_1 \oplus \pi_2$,
where π_1 is continuous and π_2 is *singular* :
every coefficient of π_2 is a.e. 0.
So we prove that if π is truly measurable, then $\pi_2 \equiv 0$.

Nonmeasurable unions

π is a singular representation.

Link to nonmeasurable unions :

Let $V = \{T \in \mathcal{U}(H) : \langle Tx, y\rangle \neq 0\}$; it's an open set.

Let $f(t) = \langle \pi(t)x, y\rangle$, then

$$A = \{t : f(t) \neq 0\} = \pi^{-1}(V)$$

is null. For any $S \subset G$, SA is measurable :

$$SA = \bigcup_{s\in S} sA = \bigcup_{s\in S} s\pi^{-1}(V) = \pi^{-1}\big(\pi(S)V\big)$$

Do such sets exist ?

Unknown ; "No" is consistent with ZFC (see part II).

Nonmeasurable unions : known results

A **Polish** space is a separable complete metric space.
Polish but not locally compact : $\mathbb{R}^{\mathbb{N}}$
Locally compact but not Polish : $\{0, 1\}^{\mathbb{R}}$

Four Poles Theorem

(Bukovsky–Brzuchowski–Cichoń–Grzegorek–Ryll-Nardzewski)

Let X be a Polish space with a Borel (completed) measure. If $\mathcal{A}$ is a point finite family of null subsets with non-null union, then there is $\mathcal{B} \subset \mathcal{A}$ with non-measurable union.

Counterexample (Fremlin)

There is a point countable family $\mathcal{A}$ of null sets in a Polish space with $\cup\mathcal{A}$ non-null such that every $\mathcal{B} \subset \mathcal{A}$ has measurable union.

Unions in a non-Polish group

Theorem 2

Let $\mathcal{A}$ be a point-finite family of null sets in a σ-compact l.c.group G. If $\cup\mathcal{A}$ is non-null and $|\mathcal{A}| \leq \mathfrak{c}$, then there is $\mathcal{B} \subset \mathcal{A}$ with $\cup\mathcal{B}$ non-measurable.

If $|\mathcal{A}| > \mathfrak{c}$ — unknown.

Is there an example in $\{0,1\}^{2^{\mathfrak{c}}}$?

Theorem is not true if G is not σ-compact ;
example in $\mathbb{R}_d \times \mathbb{R}$:

$$\mathcal{A} = \{\, (t,0) : \; t \in \mathbb{R}_d\},$$

then any sub-union of $\mathcal{A}$ has measure either 0 or ∞

Proof of Theorem 1

Reduce to a σ-compact pro-Lie subgroup ; assume π is singular.

Take $0 \neq x \in H$ and let

$$A_n = \{t : |\langle \pi(t)x, x\rangle| > 1/n\}$$

This is a null set ; n has to be chosen carefully. For any s,

$$sA_n = \{t : s^{-1}t \in A_n\} = \left\{t : |\langle \pi(s^{-1}t)x, x\rangle| > \frac{1}{n}\right\}$$
$$= \left\{t : |\langle \pi(t)x, \pi(s)x\rangle| > \frac{1}{n}\right\}$$

Find a set S such that $\pi(s)x$ are orthogonal, then
$\mathcal{A} = \{sA_n : s \in S\}$ is point-finite.
Care also that $|S| \leq \mathfrak{c}$ and SA_n is non-null (this goes together with the choice of n)... and theorem 1 is proved.

Continuity of homomorphisms

G — locally compact group,
H — any topological group,
$\pi : G \to H$ — a homomorphism.
π is measurable if $\pi^{-1}(U)$ is Haar measurable
for every open set $U \subset H$.

π is **automatically continuous** if :

H is locally compact (Kleppner, '89) ;
G and H are Polish and π is universally measurable
(Pettis, A. Weil. . .) ;
$H = \mathcal{U}(K)$, K — a Hilbert space (Theorem 1)

$H = \mathbb{R}^{\mathbb{N}}$?
non-metrizable groups ?

If G is not l.c., question is open even for two Polish groups.

Continuity of homomorphisms : theorem

Theorem 3

Under Martin's axiom, every measurable homomorphism from a l.c.g. to a topological group is continuous.

What is Martin's axiom ? A slightly weaker statement is :
if $\mathcal{A}$ is a family of null subsets of $\mathbb{R}$
and $|\mathcal{A}| < \mathfrak{c}$ then $\cup\mathcal{A}$ is null.
Also MA $\Rightarrow \mathbb{R}$ is not a union of $< \mathfrak{c}$ null sets.

MA follows from CH (Continuum hypothesis)
but is consistent with ZFC + $\neg$CH

Extra-measurable sets

G — a locally compact group

Definition

A set $A \subset G$ is extra-measurable if SA is measurable for every $S \subset G$

Example : any open set
A one-point set is never extra-measurable

If $\pi : G \to H$ is measurable and discontinuous, there is a base of nbds of 1 in H such that $\pi^{-1}(U)$ is null and extra-measurable for every U.

Theorem 4

Under MA, there are no null extra-measurable sets except $\emptyset$.

Existence of extra-measurable sets

Theorem

If G is Abelian, TFAE :

(i) There is a homomorphism $\pi : G \to H$ to a topological group H which is measurable but discontinuous ;

(ii) same with metrizable group H ;

(iii) There are null extra-measurable sets A_n, $n \in \mathbb{N}$, such that : $A_n^{-1} = A_n$; $A_{n+1}^2 \subset A_n$.

If (iii) holds, A_n can be taken as a base of nbds of 0, then the identity map is measurable but discontinuous.

Is (iii) consistent with ZFC ?
Is there an example on the real line ?..

Thank you !