Fibers of the L^∞ algebra and disintegration of measures.

Marek Kosiek

Joint work with Krzysztof Rudol

- μ - a regular probabilistic measure on a compact set X

- μ - a regular probabilistic measure on a compact set X
- X is equal to the closed support of μ

- μ - a regular probabilistic measure on a compact set X
- X is equal to the closed support of μ

Problem

Let $h \in L^\infty(\mu)$. Is it possible to define in a reasonable sense the value $h(x)$ for $[\mu]$ a.e. $x \in X$?

- Y - the spectrum of $L^{\infty}(\mu)$

- Y - the spectrum of $L^\infty(\mu)$
- $L^\infty(\mu) \sim C(Y)$ by Gelfand transform:

$$L^\infty(\mu) \ni h \to \hat{h} \in C(Y)$$

- Y - the spectrum of $L^\infty(\mu)$
- $L^\infty(\mu) \sim C(Y)$ by Gelfand transform:

$$L^\infty(\mu) \ni h \to \hat{h} \in C(Y)$$

- We define a mapping $\Pi : Y \to X$ as follows:

$$\Pi(y) = x \text{ iff } f(x) = \hat{f}(y) \text{ for all } f \in C(X)$$

- Y - the spectrum of $L^\infty(\mu)$
- $L^\infty(\mu) \sim C(Y)$ by Gelfand transform:

$$L^\infty(\mu) \ni h \to \hat{h} \in C(Y)$$

- We define a mapping $\Pi : Y \to X$ as follows:

$$\Pi(y) = x \text{ iff } f(x) = \hat{f}(y) \text{ for all } f \in C(X)$$

- There is a unique regular probabilistic measure $\tilde{\mu}$ on Y such that

$$\int_X h\, d\mu = \int_Y \hat{h}\, d\tilde{\mu}$$

for $h \in L^\infty(\mu)$

- Y - the spectrum of $L^\infty(\mu)$
- $L^\infty(\mu) \sim C(Y)$ by Gelfand transform:

$$L^\infty(\mu) \ni h \to \hat{h} \in C(Y)$$

- We define a mapping $\Pi : Y \to X$ as follows:

$$\Pi(y) = x \text{ iff } f(x) = \hat{f}(y) \text{ for all } f \in C(X)$$

- There is a unique regular probabilistic measure $\tilde{\mu}$ on Y such that

$$\int_X h\, d\mu = \int_Y \hat{h}\, d\tilde{\mu}$$

for $h \in L^\infty(\mu)$

Theorem

If $h \in L^\infty(\mu)$ then there exists an open dense subset U_h of Y with $\tilde{\mu}(U_h) = \tilde{\mu}(Y)$ such that $\hat{h}$ is constant on $\Pi^{-1}(\{x\}) \cap U_h$ for all $x \in X$.

Disintegration of measures

- X, Z - topological compact spaces

Disintegration of measures

- X, Z - topological compact spaces
- ν - a regular complex Borel measure on X

Disintegration of measures

- X, Z - topological compact spaces
- ν - a regular complex Borel measure on X
- $P : X \to Z$ - a measurable mapping

Disintegration of measures

- X, Z - topological compact spaces
- ν - a regular complex Borel measure on X
- $P : X \to Z$ - a measurable mapping

We define $P(\nu)$ on Z as follows:

$$P(\nu)(E) := \nu(P^{-1}(E))$$

for $E \subset Z$, E Borel or $[\mu]$ measurable, so that

$$\int_Z h\, dP(\nu) = \int_X (h \circ P)\, d\nu, \quad h \in C(Z).$$

Theorem

There is a measurable family of regular complex measures ν_z on Z such that $\nu = \int_Z \nu_z \, dP(\nu)$ i.e.

$$\int h d\nu = \int_Z \left(\int h(x) d\nu_z(x) \right) dP(\nu)(z) \quad \text{for } h \in C(X).$$

Each measure ν_z is supported on the fiber $P^{-1}(\{z\})$.

Sketch of the proof

Let for $f \in C(X)$

$$g_f := \frac{d(P(f\nu))}{d(P(|\nu|))}$$

Sketch of the proof

Let for $f \in C(X)$

$$g_f := \frac{d(P(f\nu))}{d(P(|\nu|))}$$

Let Y be the spectrum of $L^\infty(P(|\nu|))$. Using Banach limit we define for any $z \in Z$ a linear bounded functional of norm 1:

$$C(X) \ni f \to \Phi_z(f) := \operatorname*{Lim}_{E \in \mathcal{U}_z} \frac{1}{(\widetilde{P(|\nu|)})(E)} \int_E \widehat{g}_f \, d(\widetilde{P(|\nu|)})$$

where

$$\mathcal{U}_z := \{\Pi^{-1}(\Pi(V)) : V \subset Y, V \text{ closed-open}, \ z \in \Pi(V)\}.$$

Let for $f \in C(X)$

$$g_f := \frac{d(P(f\nu))}{d(P(|\nu|))}$$

Let Y be the spectrum of $L^\infty(P(|\nu|))$. Using Banach limit we define for any $z \in Z$ a linear bounded functional of norm 1:

$$C(X) \ni f \to \Phi_z(f) := \operatorname*{Lim}_{E \in \mathcal{U}_z} \frac{1}{\widetilde{(P(|\nu|))}(E)} \int_E \widehat{g}_f \, d(\widetilde{P(|\nu|)})$$

where

$$\mathcal{U}_z := \{\Pi^{-1}(\Pi(V)) : V \subset Y, V \text{ closed-open}, \ z \in \Pi(V)\}.$$

Hence for each $z \in Z$ there exists a regular complex Borel measure ν_z on X such that

$$\Phi_z(f) = \int f \, d\nu_z \quad \text{for} \quad f \in C(X), \quad \|\nu_z\| \leqslant 1$$

Lemma

For $a \in \Pi^{-1}(\{z\}) \cap U_{g_f}$ *we have* $\Phi_z(f) = \widehat{g}_f(a)$.

Lemma

For $a \in \Pi^{-1}(\{z\}) \cap U_{g_f}$ we have $\Phi_z(f) = \widehat{g}_f(a)$.

Then for $f \in C(X)$,

$$\int_X f\,d\nu = \int_Z g_f\,d(P(|\nu|))$$
$$= \int_Y \widehat{g}_f\,d(\widetilde{P(|\nu|)}) = \int_{Y \cap U_{g_f}} \widehat{g}_f(a)\,d(\widetilde{P(|\nu|)})(a)$$
$$= \int_{Y \cap U_{g_f}} \Phi_{\Pi(a)}(f)\,d(\widetilde{P(|\nu|)})(a) = \int_Y \Phi_{\Pi(a)}(f)\,d(\widetilde{P(|\nu|)})(a)$$
$$= \int_Z \Phi_z(f)\,d(P(|\nu|))(z) = \int_Z \left(\int f\,d\nu_z \right) d(P(|\nu|))(z).$$

Lemma

For $a \in \Pi^{-1}(\{z\}) \cap U_{g_f}$ we have $\Phi_z(f) = \widehat{g}_f(a)$.

Then for $f \in C(X)$,

$$\int_X f\, d\nu = \int_Z g_f\, d(P(|\nu|))$$
$$= \int_Y \widehat{g}_f\, d(\widetilde{P(|\nu|)}) = \int_{Y \cap U_{g_f}} \widehat{g}_f(a)\, d(\widetilde{P(|\nu|)})(a)$$
$$= \int_{Y \cap U_{g_f}} \Phi_{\Pi(a)}(f)\, d(\widetilde{P(|\nu|)})(a) = \int_Y \Phi_{\Pi(a)}(f)\, d(\widetilde{P(|\nu|)})(a)$$
$$= \int_Z \Phi_z(f)\, d(P(|\nu|))(z) = \int_Z \left(\int f\, d\nu_z \right) d(P(|\nu|))(z).$$

For ν non-negative, $h \in C(Z)$, denoting $f := h \circ P$ we get $g_f = h$, and hence $\Phi_z(f) = h(z)$ which implies that $P(\nu_z)$ is the point mass 1 measure δ_z at z, proving that ν_z is carried by $P^{-1}(\{z\})$. The general case follows almost immediately from the case of non-negative ν.

Using the disintegration formula to the result on $f \in L^\infty(\mu)$ we get:

Theorem

For each $f \in L^\infty(\mu)$ there is a family $\{\nu_x\}_{x\in X}$ of Borel regular measures on the spectrum of $L^\infty(\mu)$ such that the Gelfand transform $\hat{f}$ of f satisfies the disintegration formula

$$\int \hat{f} d\tilde{\mu} = \int_X \left(\int \hat{f}(y) d\nu_x(y) \right) d\mu(x)$$

and $\hat{f}$ is constant $[\nu_x]$ a.e. on $\Pi^{-1}(\{x\})$ for $[\mu]$ almost every $x \in X$.

Using the disintegration formula to the result on $f \in L^\infty(\mu)$ we get:

Theorem

For each $f \in L^\infty(\mu)$ there is a family $\{\nu_x\}_{x\in X}$ of Borel regular measures on the spectrum of $L^\infty(\mu)$ such that the Gelfand transform $\hat{f}$ of f satisfies the disintegration formula

$$\int \hat{f} d\tilde{\mu} = \int_X \left(\int \hat{f}(y) d\nu_x(y) \right) d\mu(x)$$

and $\hat{f}$ is constant $[\nu_x]$ a.e. on $\Pi^{-1}(\{x\})$ for $[\mu]$ almost every $x \in X$.

Remark. Roughly speaking, we can say that $\hat{f}$ is almost constant on almost every fiber $\Pi^{-1}(\{x\})$.

THANK YOU!