Boundaries for operator systems

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Boundaries for function spaces

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If $M \subset C(\overline{\mathbb{D}})$ is the disk algebra, then the Choquet boundary is $\mathbb{T}$.

Operator systems

- A (concrete) *operator system* S is a unital self-adjoint linear subspace of $B(H)$. A linear map ϕ from S to an operator system V is called *completely positive* (cp) if for all $n \in \mathbb{N}$, whenever $(s_{ij}) \in M_n(S)$ is positive, then

$$\phi^{(n)}((s_{ij})) = \left(\ \phi(s_{ij}) \ \right)$$

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  is also positive in $M_n(V)$. If ϕ is also unital, we say ϕ is ucp.
- Denote the collection of ucp maps from S to $B(H)$ by $\mathrm{UCP}(S, B(H))$. When H is finite-dimensional, these are *matrix states*.

Boundary ideals and boundary representations

Let S be a concrete operator system and let $A = C^*(S)$.

- An ideal J of A is a *boundary ideal* for S if the quotient $A \to A/J$ is completely isometric on S. A boundary ideal that contains all other boundary ideals is called the *Shilov ideal* for S.

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- An irreducible representation π of A is called a *boundary representation* for S if $\pi|_S$ extends uniquely, as a ucp map, to all of A (i.e., the only ucp extension of $\pi|_S$ is π). Denote the collection of boundary representations for S by ∂_S.

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- Boundary representations are the analogues of Choquet points in the commutative case.

Arveson's theorem

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Theorem (Arveson, 2006)

If S is a separable concrete operator system, then the Shilov ideal for S is intersection of the kernels of boundary representations. Equivalently, for every $n \in \mathbb{N}$ and $(s_{ij}) \in M_n(S)$,

$$\|(s_{ij})\| = \sup_{\pi \in \partial_S} \|\pi^{(n)}((s_{ij}))\|$$

Proof uses some deep disintegration theory.

Boundaries for operator systems

We can improve this result.

Theorem (K., 2011)

Let S be a separable operator system. Then for every $n \in \mathbb{N}$ and $(s_{ij}) \in M_n(S)$,

$$\|(s_{ij})\| = \max_{\pi \in \partial_S} \|\pi^{(n)}((s_{ij}))\|$$

In other words, the Choquet boundary is a boundary in the classical sense.

C^*-convexity

Let A be a unital C^*-algebra and let $\Lambda \subset A$.

- A (proper) *C^*-convex combination* of $\{a_1, a_2, \ldots, a_m\} \subset \Lambda$ is a sum of the form

$$a = \sum_{i=1}^{m} x_i^* a_i x_i$$

where $\{x_1, x_2, \ldots, x_m\} \subset A$ are invertible and $\sum_{i=1}^{m} x_i^* x_i = I$.

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Theorem (Morenz, 1994)

Let $\Lambda \subset M_n$ be compact and C^*-convex. Then Λ is the C^*-convex hull of its C^*-extreme points.

Example: Algebraic matricial ranges

Let $x \in B(H)$ and let $n \in \mathbb{N}$. The n^{th} *(algebraic) matricial range of* x, denoted $W^n(x)$, is

$$\{\phi(x) : \phi : B(H) \to M_n, \phi \text{ is ucp}\}$$

$W^n(x)$ is always C^*-convex.

Pure ucp maps

A ucp map $\phi : S \to B(H)$ is called *pure* if whenever $\psi : S \to B(H)$ is another cp map with $\phi - \psi$ also cp, then there exists $t \in [0, 1]$ such that $\psi = t\phi$.

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- Let $x \in B(H)$ and let $S = \text{span}\{x, x^*, I\}$. Then $\phi : S \to M_k$ is pure iff $\phi(x)$ is irreducible and C^*-extreme in $W^k(x)$.

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- Let $\phi : S \to M_1 = \mathbb{C}$ be ucp. Then ϕ is pure (i.e. a pure state) iff ϕ is a (linear) extreme point of the state space for S.

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Note: if $\phi : S \to B(H)$ is linearly extreme in $\text{UCP}(S, B(H))$, ϕ is not necessarily pure ucp.

Sketch of proof

Recall:

Theorem (K., 2011)

Let S be a separable operator system. Then for every $n \in \mathbb{N}$ and $(s_{ij}) \in M_n(S)$,

$$\|(s_{ij})\| = \max_{\pi \in \partial_S} \|\pi^{(n)}((s_{ij}))\|$$

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- Every pure ucp map (and in particular, every pure matrix state) has a pure extension to $C^*(M_n(S))$. This extension can be written as a compression of a boundary representation. So an extension of ϕ is $V^*\pi|_S V$, where $\pi \in \partial_S$.

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- Then $\|(s_{ij})\| = \|\phi((s_{ij}))\| \leq \|\pi((s_{ij}))\| = \|\sigma^{(n)}((s_{ij}))\| \leq \|(s_{ij})\|$.

Consequences

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- The fact that boundary representations form a boundary when S is separable makes it clear that in this case, all peaking representations are boundary representations.
- In contrast to the commutative case, there are operator algebras with no peaking representations. The operator algebra generated by the unilateral shift on ℓ^2 is one such example.

Strong boundary points

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- Let $\hat{A}$ be the spectrum of A, topologized with the hull-kernel topology from the primitive ideal space.
- $[\pi] \in \hat{A}$ is called a *strong boundary point* if for any open U containing $[\pi]$, there exist $n \in \mathbb{N}$ and $(s_{ij}) \in M_n(S)$ such that $\|\pi^{(n)}((s_{ij}))\| = \|(s_{ij})\|$ and $\|\sigma^{(n)}((s_{ij}))\| < \|\pi^{(n)}((s_{ij}))\|$ for all $\sigma \notin U$.

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Theorem (K., 2011)

Every Choquet point for an operator system is a strong boundary point. Thus every separable operator system has a boundary consisting of strong boundary points.

Thanks!