Uniqueness of the maximal ideal of the Banach algebra of bounded operators on $\mathbf{C([0, \omega_1])}$

joint work with Niels Jakob Laustsen

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Some background results

Theorem

For each Banach space E below the lattice of closed ideals of the Banach algebra $\mathcal{B}(E)$ is of the form:

$$\{0\} \subsetneq \mathcal{K}(E) \subsetneq \mathcal{B}(E)$$

where

- $E = \ell^2$ *(Calkin, 1941)*
- $E = c_0$ *or* $E = \ell^p$, $1 \leq p < \infty$ *(Gohberg, Markus and Feldman, 1960)*

- We identify (isomorphically) c_0 with $C([0, \omega])$.
- Bessaga and Pełczyński classified the spaces $C([0, \omega^{\omega^\alpha}]), \alpha < \omega_1$ as all the isomorphic types for the spaces $C(K)$, where K is countable.
- the Banach algebra $\mathcal{B}(C[0, \omega^\omega])$ has the unique maximal ideal consisiting of weak Banach–Saks operators
- the cases $2 \leq \alpha < \omega_1$ remain unravelled...

The work of R. Loy and G. Willis

Recall that neither the James space J_2 nor $C([0,\omega_1])$ is isomorphic to its square.

Theorem (Loy and Willis, 1989)

Let $E = J_2$ or $E = C([0,\omega_1])$. Then

- *the Banach algebra $\mathcal{B}(E)$ has an ideal $\mathcal{M}$ of codimension 1;*
- *the ideal $\mathcal{M}$ has a BRAI;*
- *each derivation from $\mathcal{B}(E)$ into a Banach $\mathcal{B}(E)$-bimodule is continuous.*

For $E = J_2$ we have $\mathcal{M} = \mathcal{W}(E)$, the ideal of weakly compact operators. If $E = C([0,\omega_1])$, then $\mathcal{M}$ has a much more complicated description. We call $\mathcal{J}$ the *Loy–Willis* ideal in honour of its discoverers.

Operators on $C([0,\sigma])$

Each ordinal σ, the ordinal interval $[0,\sigma](=\sigma+1)$ is a compact Hausdorff space which is scattered (contains no perfect subsets).

- For each $\alpha \leq \sigma$ and $T \in \mathcal{B}(C([0,\sigma]))$, the evaluation functional $f \mapsto (Tf)(\alpha)$ is continuous and has the form

$$\sum_{\beta\leq\sigma} T_{\alpha,\beta} f(\beta),$$

  where $\beta \mapsto T_{\alpha,\beta}$ is a scalar-valued, absolutely summable function defined on $[0,\sigma]$.
- We assiociate with each T the $(\sigma+1)\times(\sigma+1)$-matrix $[T_{\alpha,\beta}]_{\alpha,\beta\leq\sigma}$.
- The composition ST of operators S and T corresponds to the "usual" matrix multiplication:

$$(ST)_{\alpha,\beta} = \sum_{\gamma\leq\sigma} S_{\alpha,\gamma} T_{\gamma,\beta}, \qquad \alpha,\beta\leq\sigma.$$

Operators on $C([0,\omega_1])$

For an operator T on $C([0,\omega_1])$ and $\alpha \leq \omega_1$, we denote by r_α^T and k_α^T the α^{th} row and column of T, respectively.

Proposition (Loy and Willis)

Let T be an operator on $C([0,\omega_1])$. Then:

(i)
$$\|T\| = \sup_{\alpha\leq\omega_1}\left(\sum_{\beta\leq\omega_1}|T_{\alpha,\beta}|\right);$$

(ii) *the function k_α^T is continuous whenever $\alpha = 0$ or α is a countable successor ordinal;*

(iii) *the function k_α^T is continuous at ω_1 for each countable ordinal α;*

(iv) *the restriction of $k_{\omega_1}^T$ to $[0,\omega_1)$ is continuous, and $\lim_{\alpha\to\omega_1} k_{\omega_1}^T(\alpha)$ exists.*

The Loy–Willis ideal

Let

$$\mathcal{M} = \{T \in \mathcal{B}(C([0,\omega_1]) \colon k^T_{\omega_1} \text{ is continuous (at } \omega_1)\}$$

which is a subspace of codimension one in $\mathcal{B}(C([0,\omega_1]))$ (Prop. (iv)).

It is also an ideal of $\mathcal{B}(C([0,\omega_1]))$ as well, therefore a maximal ideal, *the Loy–Willis ideal*.

It is straightforward to verify that every operator on $C([0,\omega_1])$ not belonging to $\mathcal{M}$ has uncountably many non-zero rows and columns.

The results

Let W, X and Y be Banach spaces. An operator $T\colon X \to Y$ **fixes a copy of** W if there is an operator $U\colon W \to X$ such that the composite operator TU is an isomorphism onto its range.

If in addition we can arrange that the range of TU is complemented in Y, then we say that T **fixes a co-complemented copy of** W.

Theorem (Ka. & Laustsen, 2011)

(i) *The Loy–Willis* $\mathcal{M}$ *is the unique maximal ideal of* $\mathcal{B}(C([0,\omega_1]))$*;*

(ii) *An operator on* $C([0,\omega_1])$ *belongs to* $\mathcal{M}$ *if and only if it does not fix a co-complemented copy of* $C([0,\omega_1])$*.*

Sketch of the proof of ii)

- Take $T \notin \mathcal{M}$ and observe that it is enough to find a projection P in the ideal $\langle T \rangle$ with range isomorphic to $C([0, \omega_1])$.

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- We reduce the proof to the case $r^T_{\omega_1} = k^T_{\omega_1} = \mathbf{1}_{\{\omega_1\}}$.
- We put $\alpha_0 = \min\{\alpha \colon k^T_\alpha \neq 0\}$, $\xi_0 = \eta_0 = 0$ and

$$\xi_1 = (\sup \operatorname{supp}(k^T_{\alpha_0})) + \omega,\ \eta_1 = \Big(\sup \bigcup_{\alpha \leq \xi_1 + \omega} \operatorname{supp}(r^T_\alpha)\Big) + \omega.$$

Suppose that $\sigma > 1$ and for all $\tau < \sigma$ the numbers ξ_τ and η_τ are already chosen.

If $\sigma = \tau + 1$ for some $\tau < \omega_1$, then we define

$$\xi_\sigma = \sup\Big(\bigcup_{\alpha \leq \eta_\tau} \operatorname{supp}(k^T_\alpha)\Big) + \xi_\tau + 1,\ \eta_\sigma = \Big(\sup \bigcup_{\alpha \leq \xi_\sigma + \omega} \operatorname{supp}(r^T_\alpha)\Big) + \eta_\tau + \omega.$$

Otherwise, that is when σ is a limit ordinal, we define

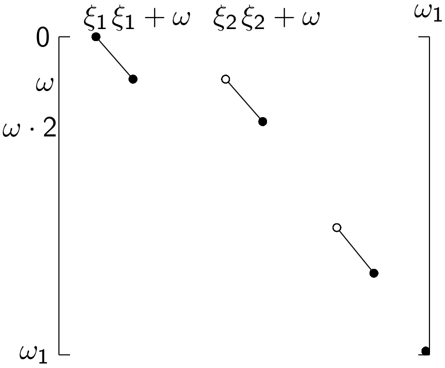
$$\xi_\sigma = \sup_{\tau < \sigma} \xi_\tau \text{ and } \eta_\sigma = \Big(\sup \bigcup_{\alpha \leq \xi_\sigma + \omega} \operatorname{supp}(r^T_\alpha)\Big) + \Big(\sup_{\tau < \sigma} \eta_\tau\Big) + \omega.$$

Sketch of the proof of ii) continued...

- Define $y = \bigcup_{\sigma<\omega_1}[\xi_\sigma, \xi_\sigma + \omega] \cup \{\omega_1\}$ and note that y is closed and order-isomorphic to $[0, \omega_1]$.

Sketch of the proof of ii) continued...

- Define $y = \bigcup_{\sigma<\omega_1}[\xi_\sigma, \xi_\sigma + \omega] \cup \{\omega_1\}$ and note that y is closed and order-isomorphic to $[0, \omega_1]$.
- Structure of the matrix of S^y:



Sketch of the proof of ii) continued...

- Put $x = \{\eta_\sigma \colon \sigma < \omega_1\}$ and

$$\varphi_x(\alpha) = \begin{cases} \eta_1 \text{ for } \alpha = 0 \\ \eta_{\sigma+1} \text{ if } \alpha \in (\eta_\sigma, \eta_{\sigma+1}] \text{ for some } \sigma \in [0, \omega_1) \\ \omega_1 \text{ for } \alpha = \omega_1 \end{cases}$$

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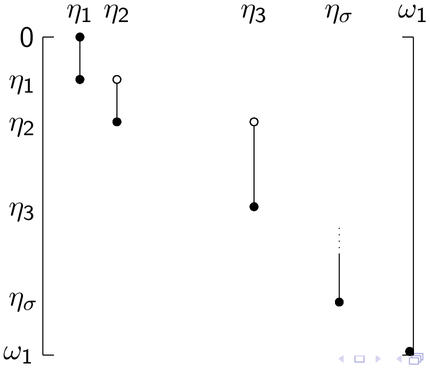
- Associate with x the composition operator $R^x : f \mapsto f \circ \varphi_x$



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- $\langle T \rangle \supseteq \langle P \rangle = \mathcal{B}(C([0, \omega_1]))$, since the identity operator factors through the range of P.

Question

- *Can we omit the adjective "co-complemented" in the statement of (ii)? More precisely, does every copy of* $C([0, \omega_1])$ *inside* $C([0, \omega_1])$ *contain a complemented copy of itself?*[a]

[a] We know that there are uncomplemented ones.

Further results on operators with separable range

It is not difficult to spot that the ideal $\mathcal{X}(C([0,\omega_1]))$ is properly contained in $\mathcal{M}$. Furthermore, we have

Theorem (Ka. & Laustsen)

The following four conditions are equivalent for an operator T on $C([0,\omega_1])$:

(b) *T factors through $C([0,\sigma])$ for some countable ordinal σ;*

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(b) *T factors through $C([0,\sigma])$ for some countable ordinal σ;*

(c) $T \in \mathscr{X}(C([0,\omega_1]))$;

(d) *T does not fix a copy of $c_0(\omega_1)$.*

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