Function Algebras Invariant under Group Actions

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Motivation: A conjecture of Arveson in operator theory.

Arveson Conjecture

H_n^2 = the n-shift space (a certain Hilbert space of holomorphic functions on B)

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Theorem (Arveson 2002): The commutators $[S_i, S_j]$ are compact.

Conjecture (Arveson 2002): The same holds for the restrictions of the S_i to any submodule of H^2_n generated by homogeneous polynomials.

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Peak point theorems (Anderson , I., Wermer (2000's))

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$n = 1$: Yes. (Wermer's maximality theorem)

$n = 2$: Yes (provided A is generated by C^1 functions). (An application of a peak point theorem)

$n \geq 3$: No. (Modification of an example of Basener)

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We will consider a different related question.

What about function algebras invariant under *all* self-homeomorphisms?

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Answer: No.

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However, the answer is yes for "nice" spaces.

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Yes for more general spaces on which self-homeomorphism group does *not* act transitively (e.g., manifolds-with-boundary, simplicial complexes).

Definition: An *indexed manifold complex* or *IM complex* is a T_2 space X and an indexed collection $\{M_\alpha\}_{\alpha \in J}$ of disjoint manifolds (without boundary) whose union is X such that

(i) J is well-ordered

(ii) for each $\alpha_0 \in J$ the set $\bigcup_{\alpha \geq \alpha_0} M_\alpha$ is closed in X

(iii) for each $\alpha_0 \in J$, each $p \in M_{\alpha_0}$ has a neighborhood N in M_{α_0} such that every self-homeomorphism of $\bigcup_{\alpha \geq \alpha_0} M_\alpha$ that is the identity outside of N extends to a self-homeomorphism of X that maps each M_α onto itself.

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As a special case we recover the following:

Theorem (Rudin 1957): If X is a compact space with no perfect subsets, then $C(X)$ is the only function algebra on X.

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Proof in two main steps (both by transfinite induction):

Step 1: Show maximal ideal space of A is X.

Step 2: Prove existence of a large supply of smooth functions and apply a general function algebra theorem about approximation of manifolds.

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Theorem (Hörmander-Wermer, etc.): Suppose $M \subset \mathbb{C}^n$ is a smooth manifold with no complex tangents and M is polynomially convex. Then $P(M) = C(M)$.

Theorem (Hörmander-Wermer, Fornæss, ... , I.): Suppose $\mathfrak{M}_A = X$ and $U \subset X$ is an open set that is an m-dimensional manifold. Suppose also that each $p \in U$ has a neighborhood V such that in some coordinate system on V

(i) $\exists f_1, \ldots, f_m \in A$ smooth on V with $df_1 \wedge \ldots \wedge df_m(p) \neq 0$, and

(ii) the functions in A smooth on V are dense in A.

Then $A \supset \{g \in C(X) : g|_{(X \setminus U)} = 0\}$.
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Are there noncommutative analogues of the theorems concerning function algebras invariant under group actions?