A Stone-von Neumann theorem over quantum groups and the convolution algebra $(T(L_2(\mathbb{G})), \triangleright)$

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Banach Algebras 2011, University of Waterloo

August 3 - 10

The Stone-von Neumann theorem

Let G be a locally compact group. The Stone-von Neumann theorem says that

$$C_0(G) \rtimes_r G \;\cong\; K(L_2(G))$$

More precisely, if $M : f \longmapsto M_f$ is the canonical representation of $C_0(G)$ on $L_2(G)$ and λ is the left regular representation of G, then (M, λ) is a *covariant* representation of $(C_0(G), G, \tau)$ (i.e., $M_{sf} = \lambda(s^{-1})M_f\lambda(s)$ for all $f \in C_0(G)$ and $s \in G$), and

$$M \rtimes \lambda : C_0(G) \rtimes_r G \longrightarrow B(L_2(G)),$$

$$f \in C_c(G, C_0(G)) \longmapsto \int_G M_{f(s)}\, \lambda(s)\, ds$$

is a faithful irreducible repn of $C_0(G) \rtimes_r G$ with range $K(L_2(G))$.

The Stone-von Neumann theorem

It seems that the name “Stone-von Neumann Theorem” can be traced back to the title of the paper “G. W. Mackey, *A theorem of Stone and von Neumann*, Duke Math. J. **16** (1949), 313 - 326.”

Cf. J. Rosenberg, *A selective history of the Stone-von Neumann theorem*, Operator algebras, quantization, and noncommutative geometry, 331-353, Contemp. Math. 365, AMS, 2004.

The Stone-von Neumann theorem

In the terminology of **actions** of locally compact quantum groups on C^*-algebras, we have

$$C^*_\lambda(G) \rtimes_r C_0(G) \;\cong\; K(L_2(G))$$

where $C^*_\lambda(G)$ is the reduced group C^*-algebra of G.

Note that in the setting of C^*-algebraic locally compact quantum groups, the C^*-algebra $C^*_\lambda(G)$ is the dual of $C_0(G)$.

We consider a possible quantum group version of this theorem.

Recall: von Neumann algebraic LCQGs

Let $\mathbb{G} = \{\mathcal{M}, \Gamma, \varphi, \psi\}$ be a von Neumann algebraic locally compact quantum group (Kustermans-Vaes 03).

That is,

- $\mathcal{M}$ is a von Neumann algebra
- $\Gamma : \mathcal{M} \longrightarrow \mathcal{M}\bar{\otimes}\mathcal{M}$ is a co-multiplication (i.e., Γ is a normal and unital $*$-homomorphism satisfying $(\Gamma \otimes id)\Gamma = (id \otimes \Gamma)\Gamma$)
- φ and ψ are respectively normal semifinite faithful left and right invariant weights on $(\mathcal{M}, \Gamma)$

Example. Let G be a locally compact group. Then

$$\mathbb{G} = (L_\infty(G), \Gamma, \varphi, \psi)$$

is a LCQG with the co-multiplication

$$\Gamma : L_\infty(G) \longrightarrow L_\infty(G) \bar{\otimes} L_\infty(G)$$

given by (the adjoint of the convolution on $L_1(G)$)

$$\Gamma(f)(s,t) = f(st) \quad (f \in L_\infty(G),\, s, t \in G)$$

and φ and ψ are the left and right Haar integrals over G

Recall: von Neumann algebraic LCQGs

For general $\mathbb{G} = \{\mathcal{M}, \Gamma, \varphi, \psi\}$,

- $$L_\infty(\mathbb{G}) := \mathcal{M}, \quad L_1(\mathbb{G}) := \mathcal{M}_*, \quad L_2(\mathbb{G}) := H_\varphi$$
- $L_1(\mathbb{G})$ is a completely contractive Banach algebra under
$$\star = \Gamma_* : L_1(\mathbb{G})\widehat{\otimes}L_1(\mathbb{G}) \longrightarrow L_1(\mathbb{G})$$
- For $L_\infty(G)$, $\star$ is the convolution on $L_1(G)$
- For $VN(G)$, $\star$ is the pointwise multiplication on $A(G)$

Recall: the reduced C^*-algebraic LCQG $C_0(\mathbb{G})$

$C_0(\mathbb{G}) \;:=\;$ the reduced quantum group C^*-algebra of $\mathbb{G}$

- $C_0(\mathbb{G})$ is a w^*-dense C^*-subalgebra of $L_\infty(\mathbb{G})$
- $\mathbb{G}$ is **compact** if $C_0(\mathbb{G})$ is unital; $\mathbb{G}$ is **discrete** if $\widehat{\mathbb{G}}$ is compact, or equivalently, $L_1(\mathbb{G})$ is unital
- $C_0(\mathbb{G})$ is two-sided introverted in $L_\infty(\mathbb{G}) = L_1(\mathbb{G})^*$
    - Recall: for a Banach algebra A, an A-submodule X of A^* is left introverted if $A^{**} \,\square\, X \subseteq X$

The quantum measure algebra $M(\mathbb{G})$

- The two Arens products on $C_0(\mathbb{G})^*$ coincide
- The co-multiplication Γ on the C^*-algebraic LCQG $C_0(\mathbb{G})$ induces directly a completely contractive multiplication $\star$ on $C_0(\mathbb{G})^*$
- All the three products on $C_0(\mathbb{G})^*$ are the same

$$M(\mathbb{G}) \;:=\; (C_0(\mathbb{G})^*, \star)$$

- $M(\mathbb{G})$ is a dual Banach algebra
- $L_1(\mathbb{G})$ is canonically a closed ideal in $M(\mathbb{G})$ via $f \mapsto f|_{C_0(\mathbb{G})}$

The LUC-space associated with $L_1(\mathbb{G})$

$$LUC(\mathbb{G}) \;:=\; \langle L_\infty(\mathbb{G}) \star L_1(\mathbb{G}) \rangle$$

- $C_0(\mathbb{G}) \subseteq LUC(\mathbb{G}) \subseteq M(C_0(\mathbb{G})) \subseteq L_\infty(\mathbb{G})$
- $LUC(\mathbb{G})^*$ is a Banach algebra; $\mathfrak{Z}_t(LUC(\mathbb{G})^*)$ is defined
- $LUC(\mathbb{G})^* = M(\mathbb{G}) \oplus C_0(\mathbb{G})^\perp$; $M(\mathbb{G}) \subseteq \mathfrak{Z}_t(LUC(\mathbb{G})^*)$
- This holds for left introverted $C_0(\mathbb{G}) \subseteq X \subseteq M(C_0(\mathbb{G}))$
- Similarly, $RUC(\mathbb{G}) := \langle L_1(\mathbb{G}) \star L_\infty(\mathbb{G}) \rangle$, $\cdots\cdots$

The convolution algebra $(T(L_2(\mathbb{G})), \triangleright)$

Let $V \in B(L_2(\mathbb{G}) \otimes L_2(\mathbb{G}))$ be the *right* multiplicative unitary of $\mathbb{G}$:

$$\Gamma(x) = V(x \otimes 1)V^* \quad (x \in L_\infty(\mathbb{G}))$$

- V induces on $B(L_2(\mathbb{G}))$ the co-multiplication

  $$B(L_2(\mathbb{G})) \longrightarrow B(L_2(\mathbb{G}))\bar{\otimes}B(L_2(\mathbb{G})), \; x \longmapsto V(x \otimes 1)V^*$$

  and hence induces on $T(L_2(\mathbb{G}))$ the multiplication

  $$\triangleright \; : T(L_2(\mathbb{G}))\widehat{\otimes}T(L_2(\mathbb{G})) \longrightarrow T(L_2(\mathbb{G}))$$

- $(T(L_2(\mathbb{G})), \triangleright)$ is a completely contractive Banach algebra

  For $\mathbb{G} = L_\infty(G)$, this algebra was introduced by Neufang

The convolution algebra $(T(L_2(\mathbb{G})), \triangleright)$

- $(T(L_2(\mathbb{G})), \triangleright) \longrightarrow L_1(\mathbb{G}),\ \omega \longmapsto \omega|_{L_\infty(\mathbb{G})}$ is a surjective and completely contractive algebra homomorphism so that

$$(L_1(\mathbb{G}), \star) \quad \cong \quad (T(L_2(\mathbb{G})), \triangleright) \,/\, L_\infty(\mathbb{G})_\perp$$

- $(T(L_2(\mathbb{G})), \triangleright)$ is seen as the right lifting convolution algebra of $L_1(\mathbb{G})$ via the right fundamental unitary V of $\mathbb{G}$
- The left lifting convolution algebra $(T(L_2(\mathbb{G})), \triangleleft)$ is defined via the left fundamental unitary W of $\mathbb{G}$

The *LUC*-space associated with $(T(L_2(\mathbb{G})), \triangleright)$

We simply use $T(L_2(\mathbb{G}))$ to denote $(T(L_2(\mathbb{G})), \triangleright)$

- Recall: $LUC(\mathbb{G}) := \langle L_\infty(\mathbb{G}) \star L_1(\mathbb{G}) \rangle$

Proposition

1. $\langle B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle = \langle L_\infty(\mathbb{G}) \triangleright T(L_2(\mathbb{G})) \rangle = LUC(\mathbb{G})$
2. $\langle B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle^* \cong LUC(\mathbb{G})^*$ as Banach algebras

c.b. right multipliers of $(T(L_2(\mathbb{G})), \triangleright)$

For $\mu \in RM(T(L_2(\mathbb{G})))$, we have $\mu^* : B(L_2(\mathbb{G})) \longrightarrow B(L_2(\mathbb{G}))$

and $\mu^*(L_\infty(\mathbb{G})) \subseteq L_\infty(\mathbb{G})$. Then

$$\Pi \; : \; RM(T(L_2(\mathbb{G}))) \longrightarrow RM(L_1(\mathbb{G})), \; \mu \longmapsto (\mu^*|_{L_\infty(\mathbb{G})})_*$$

Theorem

The map Π defines a completely isometric algebra isomorphism

$$RM_{cb}(T(L_2(\mathbb{G}))) \;\cong\; RM_{cb}(L_1(\mathbb{G}))$$

c.b. right multipliers of $(T(L_2(\mathbb{G})), \triangleright)$

The representation theorem

$$RM_{cb}(L_1(\mathbb{G})) \cong CB^{\sigma, L_\infty(\mathbb{G})}_{L_\infty(\hat{\mathbb{G}})}(B(L_2(\mathbb{G})))$$

by Junge-Neufang-Ruan (09) (Neufang-Ruan-Spronk (08) for commutative and co-commutative cases) can be formulated as

$$RM_{cb}(T(L_2(\mathbb{G}))) \cong CB^{\sigma, L_\infty(\mathbb{G})}_{L_\infty(\hat{\mathbb{G}})}(B(L_2(\mathbb{G})))$$

or

$$CB^{\sigma}_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) = CB^{\sigma, L_\infty(\mathbb{G})}_{L_\infty(\hat{\mathbb{G}})}(B(L_2(\mathbb{G})))$$

It is natural to know when we have

$$CB^{\sigma, K(L_2(\mathbb{G}))}_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) = CB^{\sigma, L_\infty(\mathbb{G})}_{L_\infty(\hat{\mathbb{G}})}(B(L_2(\mathbb{G})))$$

The *right* $(T(L_2(\mathbb{G})), \triangleright)$-module action on $B(L_2(\mathbb{G}))$

- $L_\infty(\mathbb{G})$, $C_0(\mathbb{G})$ are $(T(L_2(\mathbb{G})), \triangleright)$-bimodules. In particular,
  - $\langle C_0(\mathbb{G}) \triangleright T(L_2(\mathbb{G}))\rangle \;=\; C_0(\mathbb{G})$
  - $\langle L_\infty(\mathbb{G}) \triangleright T(L_2(\mathbb{G}))\rangle = \langle B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G}))\rangle \;=\; LUC(\mathbb{G})$

- However, $K(L_2(\mathbb{G}))$ is not a $(T(L_2(\mathbb{G})), \triangleright)$-submodule of $B(L_2(\mathbb{G}))$

Theorem

$$\langle K(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G}))\rangle \;=\; C_0(\mathbb{G})$$

- This shows the C^*-algebra $C_0(\mathbb{G})$ can be recovered from the convolution $\triangleright$. We will see more on this.

Left $T(L_2(\mathbb{G}))$-module action and regularity of $\mathbb{G}$

- For the left multiplicative unitary W of $\mathbb{G}$ on $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$,

  $$C(W) := \{(\iota \otimes \omega)(\Sigma W) : \omega \in T(L_2(\mathbb{G}))\} \subseteq B(L_2(\mathbb{G}))$$

  where Σ is the flip map on $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$.

- $\mathbb{G}$ is **regular** if $K(L_2(\mathbb{G})) = \langle C(W) \rangle$ (Baaj-Skandalis 93).

- Kac algebras, in particular, $L_\infty(G)$ and $VN(G)$, are regular.

  All compact and discrete quantum groups are regular.

- There exist non-semiregular $\mathbb{G}$ (Baaj-Skandalis-Vaes 03).

Left $T(L_2(\mathbb{G}))$-module action and regularity of $\mathbb{G}$

- $\langle K(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G}))\rangle \;=\; C_0(\mathbb{G}) \;\neq\; K(L_2(\mathbb{G}))$

We will see that $\langle T(L_2(\mathbb{G})) \triangleright K(L_2(\mathbb{G}))\rangle \;\neq\; K(L_2(\mathbb{G})),\, C_0(\mathbb{G})$

$$_{\triangleright}K(L_2(\mathbb{G})) \;:=\; \langle T(L_2(\mathbb{G})) \triangleright K(L_2(\mathbb{G}))\rangle$$

Theorem

$$_{\triangleright}K(L_2(\mathbb{G})) \;=\; U\langle C(W)\rangle U^* \;=\; \langle C_0(\mathbb{G})C_0(\widehat{\mathbb{G}})\rangle$$

where $U = \widehat{J}J$ is a unitary operator on $L_2(\mathbb{G})$

Left $T(L_2(\mathbb{G}))$-module action and regularity of $\mathbb{G}$

$$_{\triangleright}K(L_2(\mathbb{G})) \;:=\; \langle T(L_2(\mathbb{G})) \triangleright K(L_2(\mathbb{G}))\rangle$$

Corollary

1. $_{\triangleright}K(L_2(\mathbb{G}))$ is a C^*-subalgebra of $B(L_2(\mathbb{G}))$
2. $\mathbb{G}$ is regular $\iff$ $K(L_2(\mathbb{G})) \;=\; {}_{\triangleright}K(L_2(\mathbb{G}))$

Recall: action and reduced crossed product

A (continuous, left) **action** of $\mathbb{G}$ on a C^*-algebra B is a non-degenerate $*$-homomorphism

$$\alpha : B \longrightarrow M(C_0(\mathbb{G}) \otimes B)$$

with $(1 \otimes \alpha)\alpha = (\Delta \otimes \iota)\alpha$ and $\langle \alpha(B)(C_0(\mathbb{G}) \otimes 1) \rangle = C_0(\mathbb{G}) \otimes B$

Then $\langle \alpha(B)(C_0(\widehat{\mathbb{G}}) \otimes 1) \rangle \subseteq M(K(L_2(\mathbb{G})) \otimes B)$ is a C^*-algebra

$$C_0(\widehat{\mathbb{G}}) \ {}_r\!\ltimes B \ := \ \langle \alpha(B)(C_0(\widehat{\mathbb{G}}) \otimes 1) \rangle$$

Taking $(B, \alpha) = (C_0(\mathbb{G}), \Gamma)$, we have

$$C_0(\widehat{\mathbb{G}}) \ {}_r\!\ltimes C_0(\mathbb{G}) \ \subseteq \ M(K(L_2(\mathbb{G})) \otimes C_0(\mathbb{G}))$$

A Stone-von Neumann theorem over LCQGs

Theorem

1. $$C_0(\widehat{\mathbb{G}}) \,{}_r\!\ltimes C_0(\mathbb{G}) \;\cong\; {}_{\triangleright}K(L_2(\mathbb{G})) \;:=\; \langle T(L_2(\mathbb{G})) \triangleright K(L_2(\mathbb{G}))\rangle$$
2. $$C_0(\mathbb{G}) \rtimes_r C_0(\widehat{\mathbb{G}}') \;\cong\; K_{\triangleleft}(L_2(\mathbb{G})) \;:=\; \langle K(L_2(\mathbb{G})) \triangleleft T(L_2(\mathbb{G}))\rangle$$

In particular, if $\mathbb{G}$ is regular, then

$$C_0(\widehat{\mathbb{G}}) \,{}_r\!\ltimes C_0(\mathbb{G}) \;\cong\; K(L_2(\mathbb{G})) \;\cong\; C_0(\mathbb{G}) \rtimes_r C_0(\widehat{\mathbb{G}}')$$

- $$\mathbb{C} \,{}_r\!\ltimes C_0(\mathbb{G}) \;\cong\; C_0(\mathbb{G}) \;=\; \langle K(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G}))\rangle$$
- $$C_0(\mathbb{G}) \rtimes_r \mathbb{C} \;\cong\; C_0(\mathbb{G}) \;=\; \langle T(L_2(\mathbb{G})) \triangleleft K(L_2(\mathbb{G}))\rangle$$

Invariance problems, I

The representation $RM_{cb}(T(L_2(\mathbb{G}))) \cong CB^{\sigma,\, L_\infty(\mathbb{G})}_{L_\infty(\widehat{\mathbb{G}})}(B(L_2(\mathbb{G})))$ also has the following form:

$$CB^{\sigma,\, {}_\triangleright K(L_2(\mathbb{G}))}_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) \; = \; CB^{\sigma,\, C_0(\mathbb{G})}_{C_0(\widehat{\mathbb{G}})}(B(L_2(\mathbb{G})))$$

Here, $({}_\triangleright K(L_2(\mathbb{G})), T(L_2(\mathbb{G})))$ and $(C_0(\mathbb{G}), C_0(\widehat{\mathbb{G}}))$ are related by

$$\begin{aligned} {}_\triangleright K(L_2(\mathbb{G})) &= C_0(\widehat{\mathbb{G}}) \; {}_r\!\ltimes \; C_0(\mathbb{G}) \\ C_0(\mathbb{G}) &= \langle \, {}_\triangleright K(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \, \rangle \end{aligned}$$

Invariance problems, II

Theorem

T. F. A. E.

1. $$CB_{T(L_2(\mathbb{G}))}^{\sigma,\,K(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) \;=\; CB_{L_\infty(\widehat{\mathbb{G}})}^{\sigma,\,L_\infty(\mathbb{G})}(B(L_2(\mathbb{G})))$$
2. $$L_1(\mathbb{G}) \;\subseteq\; CB_{T(L_2(\mathbb{G}))}^{\sigma,\,K(L_2(\mathbb{G}))}(B(L_2(\mathbb{G})))$$
3. $$\langle T(L_2(\mathbb{G})) \triangleright K(L_2(\mathbb{G}))\rangle \;=\; K(L_2(\mathbb{G}))$$
4. $$\langle T(L_2(\mathbb{G})) \triangleright C_0(\widehat{\mathbb{G}}')\rangle \;=\; C_0(\widehat{\mathbb{G}}')$$
5. $\triangleright$ is w^*-continuous on the left
6. $\mathbb{G}$ is regular

The separate w^*-continuity of $\triangleright$

Theorem

T. F. A. E.

1. $(T(L_2(\mathbb{G})), \triangleright)$ is a dual Banach algebra
2. the product $\triangleright$ on $T(L_2(\mathbb{G}))$ is w^*-cont on the right
3. the map $T(L_2(\mathbb{G})) \longrightarrow M(\mathbb{G}), \omega \longmapsto \omega|_{C_0(\mathbb{G})}$ is w^*-w^* cont
4. $C_0(\mathbb{G}) \subseteq K(L_2(\mathbb{G}))$ canonically
5. $\mathbb{G}$ is discrete

One-sided ideal problems

Theorem

T. F. A. E.

1. $T(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G}))^{**} \subseteq T(L_2(\mathbb{G}))$
2. $LM_{cb}(T(L_2(\mathbb{G})))^c = RM_{cb}(T(L_2(\mathbb{G})))$ in $CB(B(L_2(\mathbb{G})))$
3. $\mathbb{G}$ is compact

Theorem

T. F. A. E.

1. $T(L_2(\mathbb{G}))^{**} \triangleright T(L_2(\mathbb{G})) \subseteq T(L_2(\mathbb{G}))$
2. $RM_{cb}(T(L_2(\mathbb{G})))^c = LM_{cb}(T(L_2(\mathbb{G})))$ in $CB(B(L_2(\mathbb{G})))$
3. $\mathbb{G}$ is finite (i.e., $L_\infty(\mathbb{G})$ is finite dimensional)

Strong Arens irregularity of $(T(L_2(\mathbb{G})), \triangleright)$, I

Let A be a Banach algebra. The **left Arens product** $\Box$ on A^{**} is canonically defined when A is considered as a *left* A-module:

for $a, b \in A$, $f \in A^*$, and $m, n \in A^{**}$, we have

$$\langle f \cdot a, b\rangle = \langle f, ab\rangle, \ \langle n \Box f, a\rangle = \langle n, f \cdot a\rangle, \ \langle m \Box n, f\rangle = \langle m, n \Box f\rangle$$

Similarly, the **right Arens product** $\diamondsuit$ on A^{**} is canonically defined when A is considered as a *right* A-module

- Both $\Box$ and $\diamondsuit$ extend the multiplication on A
- A is said to be **Arens regular** if $\Box$ and $\diamondsuit$ coincide

SAI of $(T(L_2(\mathbb{G})), \triangleright)$, II

- $(A^{**}, \Box)$ is a **right topological semigroup** under w^*-top:

  for any fixed $m \in A^{**}$, $n \longmapsto n\Box m$ is w^*-w^* cont.

- Similarly, $(A^{**}, \Diamond)$ is a **left topological semigroup**.

- The topological centres of $(A^{**}, \Box)$ and $(A^{**}, \Diamond)$ are

$$\mathfrak{Z}_t(A^{**}, \Box) = \{m \in A^{**} : n \longmapsto m\Box n \text{ is } w^*\text{-}w^* \text{ cont.}\}$$

$$\mathfrak{Z}_t(A^{**}, \Diamond) = \{m \in A^{**} : n \longmapsto n\Diamond m \text{ is } w^*\text{-}w^* \text{ cont.}\}$$

  called the **left** and **right** topological centres of A^{**}

SAI of $(T(L_2(\mathbb{G})), \triangleright)$, III

- $A \subseteq \mathfrak{Z}_t(A^{**}, \Box) \subseteq A^{**}$; $\quad A \subseteq \mathfrak{Z}_t(A^{**}, \Diamond) \subseteq A^{**}$

- $\mathfrak{Z}_t(A^{**}, \Box) = A^{**} \iff A$ is AR $\iff \mathfrak{Z}_t(A^{**}, \Diamond) = A^{**}$

- A is said to be left strongly Arens irregular (**left SAI**) if $\mathfrak{Z}_t(A^{**}, \Box) = A$ (Dales-Lau 05).

  Similarly, A is said to be **right SAI** if $\mathfrak{Z}_t(A^{**}, \Diamond) = A$

- left SAI $\not\Longleftrightarrow$ right SAI (Dales-Lau, Neufang, etc.)

SAI of $(T(L_2(\mathbb{G})), \triangleright)$, IV

For $x \in B(L_2(\mathbb{G}))$ and $n \in B(L_2(\mathbb{G}))^*$, we say that x supports n if $\langle n, x\rangle \neq 0$.

Lemma

Let $\mathbb{G}$ be non-compact. For any $\omega \in L_\infty(\mathbb{G})_\perp$ supported by some $x_0 \in L_\infty(\widehat{\mathbb{G}})$ and $n \in C_0(\mathbb{G})^\perp$ supported by 1 (they exist),

$$m = \omega \triangleright n \in L_\infty(\mathbb{G})^\perp \setminus T(L_2(\mathbb{G}))$$

SAI of $(T(L_2(\mathbb{G})), \triangleright)$, V

Lemma

Let $v \in [RUC(\mathbb{G}) \cup {}_{\triangleright}K(L_2(\mathbb{G}))]^{\perp}$ be supported by some $y_0 \in L_\infty(\widehat{\mathbb{G}})$. Then for all $\gamma \in T(L_2(\mathbb{G}))$ supported by 1,

$$u = v \triangleright \gamma \in L_\infty(\mathbb{G})^{\perp} \setminus T(L_2(\mathbb{G}))$$

Such functionals $v \in B(L_2(\mathbb{G}))^*$ exist if $\mathbb{G}$ is regular satisfying

$$\max\{\text{dense}(RUC(\mathbb{G})), \text{dense}(C_0(\widehat{\mathbb{G}}))\} < \text{dense}\,(L_\infty(\widehat{\mathbb{G}}))$$

which is the case if $\mathbb{G}$ is compact and infinite.

SAI of $(T(L_2(\mathbb{G})), \triangleright)$, VI

For a Banach algebras A, there are 12 natural topological centres:

$$\mathfrak{S}\mathfrak{Z}_t(A^{**}, \Box), \quad \mathfrak{S}\mathfrak{Z}_t(A^{**}, \Box)_\ell \quad \subseteq \quad \mathfrak{Z}_t(A^{**}, \Box)$$

$$\mathfrak{S}\mathfrak{Z}_t(A^{**}, \Diamond), \quad \mathfrak{S}\mathfrak{Z}_t(A^{**}, \Diamond)_r \quad \subseteq \quad \mathfrak{Z}_t(A^{**}, \Diamond)$$

$$\mathfrak{S}\mathfrak{Z}_t(\langle A^*A\rangle^*) \quad \subseteq \quad \mathfrak{Z}_t(\langle A^*A\rangle^*), \quad \mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond$$

$$\mathfrak{S}\mathfrak{Z}_t(\langle AA^*\rangle^*) \quad \subseteq \quad \mathfrak{Z}_t(\langle AA^*\rangle^*), \quad \mathfrak{Z}_t(\langle AA^*\rangle^*)_\Box$$

Various Arens irregularity properties and their interrelationships have been be studied. We found that these new topological centres happen to be the hidden pieces that are responsible for some asymmetry phenomena occurred in topological centre problems.

Theorem

T. F. A. E.

1. $(T(L_2(\mathbb{G})), \triangleright)$ is SAI
2. $\mathfrak{Z}_t(T(L_2(\mathbb{G}))^{**}, \Diamond) = T(L_2(\mathbb{G}))$
3. $\mathfrak{Z}_t(T(L_2(\mathbb{G}))^{**}, \Diamond) = \mathfrak{S}\mathfrak{Z}_t(T(L_2(\mathbb{G}))^{**}, \Diamond)$
4. $\mathbb{G}$ is finite

Recall: AR and SAI are opposite properties

For a Banach algebra A, we have

$$A \subseteq \mathfrak{Z}_t(A^{**}, \Diamond) \subseteq A^{**}$$

$$A \text{ is AR} \iff \mathfrak{Z}_t(A^{**}, \Diamond) = A^{**}$$

$$A \text{ is right SAI} \iff \mathfrak{Z}_t(A^{**}, \Diamond) = A$$

Arens regularity of $(T(L_2(\mathbb{G})), \triangleright)$

For a class of *co-amenable* locally compact quantum groups $\mathbb{G}$ (e.g., $\mathbb{G} = L_\infty(G)$, or $VN(G)$ with G amenable), we have

$$L_1(\mathbb{G}) \text{ is Arens regular} \iff \mathbb{G} \text{ is finite}$$

Note that if $(T(L_2(\mathbb{G})), \triangleright)$ is Arens regular, then so is $L_1(\mathbb{G})$, since $L_1(\mathbb{G})$ is a quotient algebra of $(T(L_2(\mathbb{G})), \triangleright)$

So, for this class of $\mathbb{G}$, we have

$$(T(L_2(\mathbb{G})), \triangleright) \text{ is Arens regular} \iff \mathbb{G} \text{ is finite}$$

Therefore, as seen below, we get a natural class of Banach algebras associated with LCQGs, for which AR and SAI are surprisingly equivalent.

AR and SAI of $(T(L_2(\mathbb{G})), \triangleright)$ are often equivalent

Theorem

Let $\mathbb{G}$ be a co-amenable locally compact quantum group such that

either $\mathbb{G}$ is commutative,

or $\mathbb{G}$ is co-commutative,

or $\mathbb{G}$ is amenable with $L_1(\mathbb{G})$ separable.

T. F. A. E.

1. $(T(L_2(\mathbb{G})), \triangleright)$ is strongly Arens irregular
2. $(T(L_2(\mathbb{G})), \triangleright)$ is Arens regular
3. $\mathbb{G}$ is finite

Regular quantum groups and regular semigroups

As above, $(T(L_2(\mathbb{G})), \triangleright)$ is simply denoted by $T(L_2(\mathbb{G}))$.

- The **topological centres** of $T(L_2(\mathbb{G}))$ can be defined:

$$\mathfrak{Z}_t^{(r)}(T(L_2(\mathbb{G}))) = \{\gamma \in T(L_2(\mathbb{G})) : \omega \mapsto \omega \triangleright \gamma \text{ is } w^*\text{-cont}\}$$

$$\mathfrak{Z}_t^{(\ell)}(T(L_2(\mathbb{G}))) = \{\gamma \in T(L_2(\mathbb{G})) : \omega \mapsto \gamma \triangleright \omega \text{ is } w^*\text{-cont}\}$$

Proposition

1. $T(L_2(\mathbb{G}))$ is right regular $\iff$ $\mathbb{G}$ is regular
2. $T(L_2(\mathbb{G}))$ is regular $\iff$ $\mathbb{G}$ is discrete

The commutative and co-commutative cases

- For all $\mathbb{G}$, $L_\infty(\mathbb{G})_\perp \subseteq \mathfrak{Z}_t^{(r)}(T(L_2(\mathbb{G}))) \neq \{0\}$

For $\mathfrak{Z}_t^{(\ell)}(T(L_2(\mathbb{G})))$, we have the following dichotomy proposition.

Proposition

Let $\mathbb{G} = L_\infty(G)$ or $VN(G)$. Then

$$\mathfrak{Z}_t^{(\ell)}(T(L_2(\mathbb{G}))) = \begin{cases} T(L_2(\mathbb{G})) & \text{if } \mathbb{G} \text{ is discrete} \\ \{0\} & \text{if } \mathbb{G} \text{ is non-discrete} \end{cases}$$