A metric version of projectivity for normed spaces and modules

A. Ya. Helemskii
Faculty of Mechanics and Mathematics
Moscow State University
Moscow 119992 Russia

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The talk consists of 3 parts under common title "metrically projective normed modules"

I. We introduce the notion of metrically projective normed module. Then we characterize metrically projective normed spaces, that is modules in the case of the simplest algebra $\mathbb{C}$, as the spaces $l_1^0(M)$, the normed subspaces of $l_1(M)$, consisting of finitely supported functions.

II. We consider algebras of the next degree of complication after $\mathbb{C}$: sequence algebras, and describe metrically projective normed and Banach modules over them within some wide class of modules.

III. We show that the notion of a metrically projective, as well as of a metrically injective normed module, are but particular manifestations of some general-categorical scheme. (Another special cases are most known "relative" notions, many years taken as standard, and also some new reasonable notions.) In particular, we are able to speak about (categorically) free and cofree normed modules and apply them for our problems.

I. In functional analysis there are several different approaches to the notion of a projective module. All of them can be put in the following frame-work. Consider the category $\mathbf{A} - \mathbf{mod}$ of left normed modules over a fixed normed algebra A and their bounded morphisms or, if you wish, some its distinguished subcategory, say $\mathcal{H}$. Then, to some $P \in \mathcal{H}$, we have a standard morphism functor

$$ {}_A\mathbf{h}(P, ?) : \mathbf{A} - \mathbf{mod} \text{ (or } \mathcal{H}) \to \mathbf{Nor} : X \mapsto_A \mathbf{h}(P, X) \text{ etc.} $$

The kind of projectivity of P depends on what this functor performs with this or that class of epimorphisms in $\mathbf{A} - \mathbf{mod}$ or $\mathcal{H}$. Most important, apparently, are 4 versions of the projectivity. Two of them are formulated in terms of the norm topology of a module in question; these are what we call just projectivity (or relative projectivity), the most known version, and also the so-called topological projectivity. I do not discuss these two types, and I concentrate on two new types that take into account on the exact value of the norm. (Interest to them was stimulated by some questions of operator space theory). Namely, we call P *metrically projective*, if the functor

${}_A\mathbf{h}(P,?)$ preserves the property of an operator to be a strict coisometry, and extremely projective, if our functor preserves the property of an operator to be (just) coisometry. Here by *strict coisometry* we mean an operator which takes the closed unit ball of the domain space onto the closed unit ball of the range space, and by coisometry (they say also "quotient map") an operator that behaves similarly with open balls. (Recall that the Hahn/Banach Theorem claims, in equivalent formulation, that the adjoint to an isometry is a strict coisometry. This will be crucial to what will follow.) Today I speak mostly about the metric projectivity, which is apparently better from the categorical viewpoint and more manageable.

In the traditional form of liftings, the relevant definition sounds as follows: a normed A-module P is called metrically projective with respect of the class $\mathcal{H}$, if, for every strict coisometry $\tau : Y \to X$, where $X, Y \in \mathcal{H}$, and every bounded morphism $\varphi : P \to X$, there exists a lifting of φ across τ (that is a morphism, making the diagram

$$\begin{tikzcd} & Y \arrow[d, "\tau"] \\ P \arrow[ur, "\psi"] \arrow[r, "\varphi"'] & X \end{tikzcd}$$

commutative) such that $\|\psi\| = \|\varphi\|$.

After the definition is given, the first natural question is: what can be said about metrically projective modules in the simplest case of normed spaces, that is when $A = \mathbb{C}$.

Theorem. *Metrically projective normed spaces are exactly $l_1^0(M)$.*

(As we shall see later, this statement just means that in the case of normed spaces the metric projectivity coincides with the metric freedom.)

The proof consists of several steps. Note that our definitions of projectivity that were given above for normed (= non-complete) modules, have obvious analogues for Banach (= complete) modules. So, first, we consider the Banach case and prove that metrically projective Banach spaces are $l_1(M)$. Well, we have well known Grothendieck Theorem of 1955, characterizing those spaces. However, we can not use it as it is formulated: it tells about another kind of projectivity, the extreme projectivity, defined, in brief terms, above. But we essentially use two ingredients of his proof. Indeed, we begin with the proposition that *every metrically projective Banach space is metrically flat*, that is $E\widehat{\otimes}? : \mathbf{Ban} \to \mathbf{Ban}$, the completed projective tensor product functor with E as the given space, preserves isometries. In our case, by virtue of the above-mentioned interpretation of the Hahn/Banach Theorem, it is easy corollary of the law of adjoint associativity. After this we shoot from the first cannon, provided by Grothendieck: a metrically flat Banach space is necessarily of the form $L_1(\Omega, \mu)$, where (Ω, μ) is a measure space. Further, metrically projective spaces are retracts of free spaces, that is, in our case, of spaces $l_1(M)$, where M is an index set. (It can be easily seen directly, but later we shall show that it is a particular case of some general-categorical observation). Therefore our $L_1(\Omega, \mu)$ can be isometrically embedded into some $l_1(M)$. Now again we turn to Grothendieck: according to him, in such a case μ is necessarily discrete, and we are done.

To apply this to the non-completed case, we use the following observation: if a normed A-module X (for a moment, A is arbitrary) is metrically projective, then its completion $\overline{X}$ is metrically projective *as a Banach module.*

Thus we see that our given metrically projective space, say F, is a dense normed subspace of some $l_1(M)$.

This form approximately a half of our proof. In the second half we must show that our F contains all $l_1^0(M)$ and nothing more.

One way to do it is to study extreme points of the closed unit ball $\bigcirc_F$ of F. Here I restrict myself to say that we observe a certain "dichotomy": *for a vector e from the natural basis of $l_1^0(M)$ (formed by the "δ-functions") either e belongs to F, or e is a limit point of extreme points of $\bigcirc_F$ that are not multiples of e.*

II. In the second part of the talk we turn to normed algebras of the next degree of complication after $\mathbb{C}$. We mean algebras of functions on discrete sets. For the better transparency, we shall speak here of certain algebras, consisting of sequences. We call such an algebra, say A, a *normed sequence algebra*, if it satisfies two natural conditions: (i) A contains, as a dense subset, all finite sequences, and (ii) the norm of the "elementary" sequence $\mathbf{p}^n := (..., 0, 1, 0, ...)$, with 1 on the n–th place, is 1.

For these A we distinguish, within $\mathbf{A} - \mathbf{mod}$, a certain class

$\mathcal{H}$ and describe metrically projective modules with respect to this class. (Note that a similar, up to some nuances, description takes place also for extremely projective modules). This is done in the normed case as well as in the Banach case.

The mentioned class $\mathcal{H}$ consists of the so-called non-degenerate homogeneous normed A-modules. "Non-degenerate" means, as usual, that the linear span of all outer products is dense in our module. To define homogeneity, let us observe that every module X over a sequence algebra gives rise to its subspaces $X_n := \{\mathbf{p}^n \cdot x; x \in X\}$, the so-called coordinate subspaces. Vectors $\mathbf{p}^n \cdot x$, denoted by x_n, we call coordinates of x. We say that X is *homogeneous*, if, for every $x, y \in X$, the inequality $\|x_n\| \leq \|y_n\|$ for all n implies $\|x\| \leq \|y\|$. (Speaking informally, this means that the norm of a vector depends only on norms of its coordinates).

In Banach case we speak, of course, about modules in $\mathcal{H}$ that are complete. Their class is denoted by $\overline{\mathcal{H}}$.

In the normed case the relevant description is as follows.

Theorem. *A module $X \in \mathcal{H}$ is extremely projective with respect to $\mathcal{H}$ if and only if it satisfies the following two conditions:*

(i) For every $n \in \mathbb{N}$, the n–th coordinate subspace X_n is $l_1^0(M_n)$ for some index set M_n

(ii) for every $x \in X$ its coordinates x_n are 0 for all sufficiently large n.

As a corollary, the Banach counterpart of the formulated theorem is

Theorem. *A module $X \in \overline{\mathcal{H}}$ is extremely projective with respect to $\overline{\mathcal{H}}$ if and only if for every $n \in \mathbb{N}$ the n–th coordinate subspace X_n is $l_1(M_n)$ for some index set M_n.*

Note that in both theorems the answer depends not on the norm on the whole module but only on the norms of its coordinate subspaces. In particular, all Banach non-degenerate homogeneous modules, consisting of sequences, are extremely projective within the class of Banach non-degenerate homogeneous modules. However, neither of them, when it is infinite-dimensional, is extremely projective within the class of all normed non-degenerate homogeneous modules. On the other hand, submodules of these modules, consisting of finite sequences, are extremely projective within the latter class.

As to the proof, I shall only explain, why in the non-complete case all vectors in X have to be of finite type, that is have finite sequences of coordinates.

Take an arbitrary $X \in \mathcal{H}$ and denote by $c_0(X)$ the set of all sequences $x := (..., x_n, ...); x_n \in X_n$ such that $\|x_n\|$ tends to 0. We endow it with the coordinate-wise operations, including the outer multiplication, and with the norm $\|x\| := \max\{\|x_n\|; n \in \mathbb{N}\}$. Obviously, we get an A-module from $\mathcal{H}$. Denote by $\mathbf{in} : X \to c_0(X)$ the map, taking x to the sequence of its coordinates; clearly, it is a contractive morphism of A-modules.

Now suppose that not all vectors of X are of finite type.

To achieve our goal, we construct a module $\mathbf{Y} \in \mathcal{H}$ and a strict coisometry $\tau : \mathbf{Y} \to c_0(X)$, such that there is no even bounded ψ, making the diagram

$$\begin{tikzcd} & \mathbf{Y} \arrow[d, "\tau"] \\ X \arrow[ur, "\psi"] \arrow[r, "\mathbf{in}"'] & c_0(X) \end{tikzcd}$$

commutative.

With this aim, for every $x \in c_0(X)$ we denote by $\bar{x} \in c_0(X)$ the sequence $(..., \bar{x}_n, ...)$ where $\bar{x}_n := x_n$, if $\|x_n\| \geq 1$, or if $x_n = 0$, and $\bar{x}_n := \|x_n\|^{-\frac{1}{2}} x_n$, if $0 < \|x_n\| < 1$. After this, denote by Y_x the submodule of $c_0(X)$, algebraically generated by $\bar{x}$ (that is $Y^x := \{a \cdot \bar{x}; a \in A_+\}$, where A_+ is the unitization of A) and take, as $\mathbf{Y}$, the pure algebraic direct sum of Y^x for all $x \in X$.

Now we endow $\mathbf{Y}$ with a suitable norm. Namely, for $\mathbf{y} \in \mathbf{Y}$ with "components" $y^x \in Y^x; x \in c_0(X)$ (we remember that only finite set of those differs from 0) we put $\|\mathbf{y}\| := \max\{\sum_x \|y^x\|; x \in c_0(X)\}$. Evidently, $\mathbf{Y}$ becomes a module from $\mathcal{H}$.

As to the morphism $\tau : \mathbf{Y} \to c_0(X)$, we can define it by indicating its action on each direct summand in $\mathbf{Y}$. Namely, on Y^x, since it is generated by $\bar{x}$, our τ is well defined by taking this $\bar{x}$ to x. Since $\|\bar{x}\| = \|x\|$, we easily see that τ is a strict coisometry. But the crucial observation is that for every x, which is not of finite type, the set of numbers $\|\bar{x}_n\|/\|x_n\|$ is not bounded. Using this, it is not hard to see that every morphism ψ, making the diagram above commutative, can not be bounded.

III. In the concluding part of the talk we do some abstract nonsense. We show that the metric projectivity can be represented as a particular case of a certain general-categorical scheme. (We could show that another particular case is what is well known as “just” projectivity, and there are also some other instructive examples).

Let $\mathcal{K}$ and $\mathcal{L}$ be two arbitrary categories, connected by a faithful covariant functor $\Box : \mathcal{K} \to \mathcal{L}$. Speaking informally, we consider $\mathcal{K}$ as our main category, and a functor above as an additional structure on $\mathcal{K}$. We shall call a morphism (necessarily epimorphism) τ in $\mathcal{K}$ *admissible*, if $\Box(\tau)$ is a retraction in $\mathcal{L}$. Then we call an object $P \in \mathcal{K}$ *projective* (or, to be precise, $\Box$-projective), if the standard morphism functor $\mathbf{h}_{\mathcal{K}}(P, ?) : \mathcal{K} \to \mathbf{Sets} : X \mapsto \mathbf{h}_{\mathcal{K}}(P, X)$ etc. takes admissible epimorphisms to surjective maps.

To put in this scheme the metric projectivity, we set $\mathcal{K} := \mathbf{A} - \mathbf{mod_1}$, the category with the same objects as $\mathbf{A} - \mathbf{mod}$ but with only contractive module morphisms as categorical morphisms. Further, we take $\mathcal{L} := \mathbf{Sets}$, and as $\Box$ the functor $\bigcirc : \mathbf{A} - \mathbf{mod_1} \to \mathbf{Sets}$, taking a normed module X to its closed unit ball $\bigcirc_X$. It is almost immediate that in our case admissible morphisms are exactly strictly coisometric morphisms, and $\bigcirc$-projective objects are exactly metrically projective modules.

What if one wishes to introduce, together with projective objects or instead of them, injective objects ? The general scheme works as follows. For our $\mathcal{K}$, consider (generally speaking, another than $\mathcal{L}$) auxiliary category $\mathcal{M}$ and a covariant functor $\boxdot : \mathcal{K} \to \mathcal{M}$. We call a morphism (necessarily monomorphism) ι in $\mathcal{K}$ *admissible*, if $\boxdot(\iota)$ is a coretraction in $\mathcal{M}$. Then we call an object $J \in \mathcal{K}$ *injective* (to be precise, $\boxdot$-injective), if the standard contravariant morphism functor $\mathbf{h}_{\mathcal{K}}(?, J) : \mathcal{K} \to \mathbf{Sets} : X \mapsto \mathbf{h}_{\mathcal{K}}(X, J)$ etc. takes admissible monomorphisms to (again!) surjective maps.

To define metrically injective normed modules, we suggest to consider, as $\boxdot$, the functor $\bigodot : \mathbf{A}-\mathbf{mod_1} \to \mathbf{Sets}^{op}$ (or, equivalently, a contravariant functor from $\mathbf{A}-\mathbf{mod_1}$ to $\mathbf{Sets}$), taking X to $\bigcirc_{X^*}$, and a contractive morphism of A-modules to the respective restriction of its adjoint to unit balls. One can easily see that admissible monomorphisms are exactly the isometric morphisms, and $\bigodot$-injective objects are exactly those with the "Hahn/Banach property" ("metric injective property", as they say in Banach space geometry). So, it is natural to call these modules metrically injective.

Now we arrive to the most essential part: freedom and cofreedom. Suppose that, in the general-categorical context, $\square$ has a left adjoint functor, a certain $\mathbf{Fr} : \mathcal{L} \to \mathcal{K}$. In this situation we call $\mathbf{Fr}$ *freedom functor* (or, if we want to play a precision, $\square$-freedom functor), and the objects of the form $\mathbf{Fr}(M); M \in \mathcal{L}$ *free objects*. In a similar way, if $\boxdot$ has a right adjoint functor $\mathbf{Cfr} : \mathcal{M} \to \mathcal{K}$, we call the latter *cofreedom functor* and speak about *cofree objects*.

The practical use of the (co)freedom is provided by the following easy, and actually well known, fact: *if the freedom (resp., cofreedom) functor does exist, then an object in $\mathcal{K}$ is projective (resp., injective), if and only if it is retract of some free (resp., cofree) object.*

Note that if $\square$ and $\boxdot$ are connected in a certain way then the existence of the freedom functor implies that of the cofreedom functor, and we have a method to construct cofree objects from free objects. This is just what happens in our "metric" case.

For normed modules things behave perfectly. Both functors, freedom and cofreedom, do exist. Freedom acts as $M \mapsto A_+ \underset{p}{\otimes} l_1^0(M)$, where A_+ is the unitization of A, $\underset{p}{\otimes}$ denotes the non-completed projective tensor product, and the outer multiplication is well defined by $a \cdot (b \otimes x) = ab \otimes x$. Cofreedom acts as $M \mapsto \mathcal{B}(A_+, l_\infty(M))$, the space of all bounded operators between A_+ and $l_\infty(M)$, with $(a \cdot T)(b) := T(ba)$. I stress, that these are not "empirical" definitions, but propositions – despite, of course, very easy.

In the hindsight, we see that the theorem, discussed in the first part of the talk, means that all metrically projective normed spaces are free. Note that it follows from what was said above that there are more metrically injective spaces than cofree spaces. And, needless to say, in the case of modules over general normed algebras, free modules form only a part of projective modules.

Thank you.