Operator corona problems for function algebras

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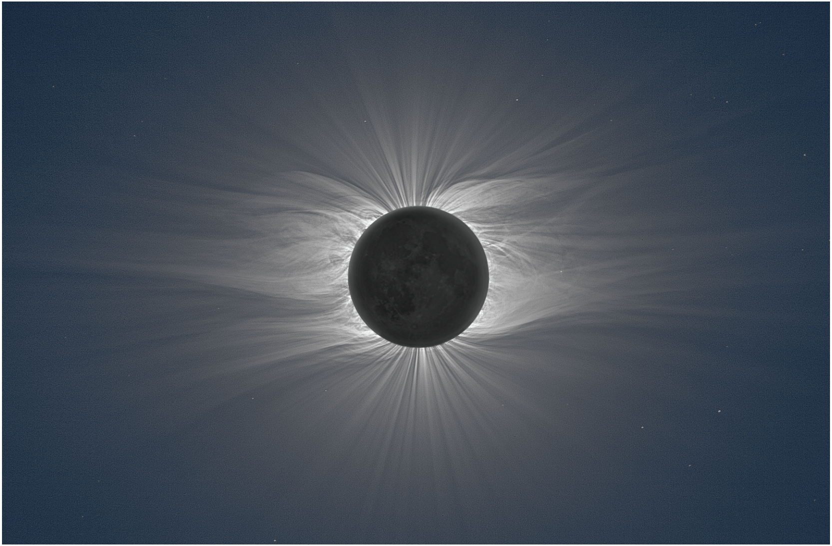

Let H^∞ denote the bounded, analytic functions on the unit disk $\mathbb{D}$.

Theorem (Carleson 1962)

Suppose $f_1, \ldots, f_n \in H^\infty$ satisfy

$$\sum_{i=1}^{n} |f_i(z)|^2 \geq \delta^2 > 0, \; z \in \mathbb{D}.$$

Then there are functions $g_1, \ldots, g_n \in H^\infty$ such that $\sum_{i=1}^{n} f_i g_i = 1$.

The Toeplitz corona theorem

Theorem (Arveson-1975; Schubert-1978)

Suppose $f_1, \ldots, f_n \in H^\infty$ *satisfy*

$$\sum_{i=1}^{n} T_{f_i} T_{f_i}^* \geq c^2 I.$$

Then there are functions $g_1, \ldots, g_n \in H^\infty$ *so that*

$$\sum_{i=1}^{n} f_i g_i = 1, \text{ and } \|[T_{g_i}, \ldots, T_{g_n}]^T\| \leq c^{-1}.$$

Suppose $\mathcal{A}$ is an operator algebra and $A_1, \ldots, A_n \in \mathcal{A}$ with $[A_1, \ldots, A_n]$ right invertible.

Operator corona problems

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There are $B_1, \ldots, B_n \in B(H)$ such that $[A_1, \ldots, A_n] \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} = I_H$.

For example, take $B_i = A_i^* \left(\sum_{i=1}^n A_i A_i^* \right)^{-1}$.

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For example, take $B_i = A_i^* \left(\sum_{i=1}^n A_i A_i^*\right)^{-1}$.
Can we find B_i in $\mathcal{A}$?

Operator corona theorem for nest algebras

Suppose $\{P_m\}_{m\geq 0}$ is an increasing sequence of projections tending strongly to I_H and let $\mathcal{A} := \mathrm{Alg}\{P_m\}_{m\geq 0}$.

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Theorem (Arveson-1975)

Suppose $A_1, \ldots, A_n \in \mathcal{A}$ satisfy

$$\sum_{k=1}^{n} A_k P_m A_k^* \geq c^2 P_m \text{ for every } m \geq 0.$$

Then there are $B_1, \ldots, B_n \in \mathcal{A}$ such that

$$\sum_{k=1}^{n} A_k B_k = I_H$$

If $\mathcal{B}$ is an algebra of operators and h a vector, let $\mathcal{B}[h] := \overline{\text{span}\{Bh : B \in \mathcal{B}\}}$.

Subalgebras of H^∞

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$\mathcal{B}[h] := \overline{\text{span}\{Bh : B \in \mathcal{B}\}}$.

Theorem (Raghupathi-Wick 2010)

Suppose $\mathcal{A}$ is a unital, weak-closed subalgebra of H^∞ and $f_1, \ldots, f_n \in \mathcal{A}$ satisfy*

$$\sum_{i=1}^{n} T_{f_i} P_L T_{f_i}^* \geq c^2 P_L$$

for every L of the form $\mathcal{A}[h]$ where h is an outer function. Then there are $g_1, \ldots, g_n \in \mathcal{A}$ so that

$$\sum_{i=1}^{n} f_i g_i = 1 \text{ and } \|[T_{g_1}, \ldots, T_{g_n}]^T\| \leq c^{-1}$$

The bidisk

Theorem (Amar 2003; Trent-Wick 2008)

Suppose $f_1, \dots, f_n \in H^\infty(\mathbb{D}^2)$ *satisfy*

$$\sum_{i=1}^{n} T^{\nu}_{f_i} (T^{\nu}_{f_i})^* \geq c^2 I_\nu$$

for every absolutely continuous measure ν *on* $\mathbb{T}^2$*. Then there are functions* $g_1, \dots, g_n \in H^\infty(\mathbb{D}^2)$ *so that*

$$\sum_{i=1}^{n} f_i g_i = 1, \text{ and } \|[T_{g_i}, \dots, T_{g_n}]^T\| \leq c^{-1}.$$

Let H be a Hilbert space of $\mathbb{C}$-valued functions on a set X. If the functionals $h \mapsto h(x)$ are bounded, then we call H a **reproducing kernel Hilbert space** (RKHS).

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The Hardy space H^2 on the unit disk $\mathbb{D}$ is the canonical example.
Its kernel is the Szegö kernel

$$k_w(z) = \frac{1}{1 - z\overline{w}}.$$

Let $\Omega \subset \mathbb{C}^d$ be a bounded domain and let μ be Lesbesgue measure on $\mathbb{C}^n$.

Example (Bergman space)

$L^2_a(\Omega) := \left\{ f \text{ homomorphic on } \Omega : \int_\Omega |f|^2 d\mu < \infty \right\}$ is a RKHS.

The kernel for $L^2_a(\mathbb{D})$ is $k_w(z) = \frac{1}{(1-\overline{w}z)^2}$.

Examples of reproducing kernel Hilbert spaces

Let $\mathbb{B}_d$ be the unit ball of $\mathbb{C}^d$ (we allow $d = \infty$).

Example (Drury-Arveson space)

H^2_d is the closure of d-variable polynomials on $\mathbb{B}_d$ with kernel

$$k^d_w(z) = \frac{1}{1 - \langle z, w \rangle_{\mathbb{C}^d}}$$

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- H^2_d is an excellent multivariable analogue of H^2
- Many function spaces embed into H^2_∞ in a natural way (Dirichlet space, Sobolev-Besov spaces).

Multiplier Algebras

The *multiplier algebra* of a RKHS H is

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is automatically bounded by the closed graph theorem.
We always have

$$M_f^* k_x = \overline{f(x)} k_x.$$

Examples of multiplier algebras

- $M(H^2((\mathbb{D}))) = H^\infty(\mathbb{D})$
  $M(H^2(\mathbb{D}^2)) = H^\infty(\mathbb{D}^2)$
  $M(L^2_a(\Omega)) = H^\infty(\Omega)$

Examples of multiplier algebras

- $M(H^2((\mathbb{D}))) = H^\infty(\mathbb{D})$
  $M(H^2(\mathbb{D}^2)) = H^\infty(\mathbb{D}^2)$
  $M(L_a^2(\Omega)) = H^\infty(\Omega)$
- $M(H_d^2) \subset H^\infty(\mathbb{B}_d)$ is the unital algebra generated by multiplication by coordinates:

  $$M(H_d^2) = \mathrm{Alg}(M_{z_1}, \ldots, M_{z_d})$$

  $[M_{z_1}, \ldots, M_{z_d}]$ is the model for row contractions (Arveson 1998).

The Toeplitz corona theorem for Drury-Arveson space

Theorem (Ball-Trent-Vinnikov 2002)

Suppose $f_1, \dots, f_n \in \mathcal{M}(H^2_d)$ *satisfy*

$$\sum_{i=1}^{n} M_{f_i} M_{f_i}^* \geq c^2 I.$$

Then there are functions $g_1, \dots, g_n \in \mathcal{M}(H^2_d)$ *so that*

$$\sum_{i=1}^{n} f_i g_i = 1.$$

and $\|[M_{g_i}, \dots, M_{g_n}]^T\| \leq c^{-1}$.

Invariant subspaces

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Invariant subspaces

- Suppose $\mathcal{A}$ is *any* unital, weakly closed algebra of multipliers on H.
- Every invariant subspace L of $\mathcal{A}$ is also a RKHS, with kernel function $k_\lambda^L := P_L k_\lambda$.
- For $f \in \mathcal{A}$ and $g \in L$ we have $fg \in L$. Call this multiplication operator M_f^L.

Our approach

Suppose $f_1, \ldots, f_n \in \mathcal{A}$ and let

$$F := [f_1, \ldots, f_n]; \;\; M_F := [M_{f_1}, \ldots, M_{f_n}] : H^{(n)} \to H.$$

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$$\left[\left(\langle F(\lambda_i)^*, F(\lambda_j)^* \rangle - c^2 \right) \langle k_{\lambda_i}, k_{\lambda_j} \rangle \right]_{i,j=1}^{k} \geq 0.$$

Our approach

The same observation is true for

$$\sum_{i=1}^{n} M_{f_i}^L (M_{f_i}^L)^* \geq c^2 I_L$$

as well, with k^L instead of k.

Our approach

Let $E = \{\lambda_1, \dots, \lambda_k\} \subset X$.
Suppose the condition

$$\left[\left(\langle F(\lambda_i)^*, F(\lambda_j)^* \rangle - c^2 \right) \langle k^L_{\lambda_i}, k^L_{\lambda_j} \rangle \right]_{i,j=1}^{k} \geq 0, L \in \mathcal{L} \qquad (1)$$

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- $\|M_{G^E}\| \leq c^{-1}$.

Our approach

This solves the operator corona problem for $\mathcal{A}$!

- The set of all such G^E are contained in a weak* compact subset.
- Point evaluation is weak*-continuous for $\mathcal{A}$.
- Thus, the G^E accumulate at some G which satisfies $\|M_G\| \leq c^{-1}$ and $\sum_{i=1}^n f_i g_i = 1$.

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Definition

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There is a contractive column multiplier $M_G = [M_{g_1}, \dots, M_{g_n}]^T$ with $g_i \in \mathcal{A}$ such that $\langle G(\lambda_i), v_i\rangle_{\mathbb{C}^n} = w_i$ for each i if and only if

$$\left[\left(\langle v_i, v_j \rangle_{\mathbb{C}^n} - w_i \overline{w_j} \right) \langle k^L_{\lambda_i}, k^L_{\lambda_j} \rangle_H \right]_{i,j=1}^k, \; L \in \mathcal{L}$$

is positive semidefinite.

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is positive semidefinite.

When $F(\lambda_i)^* = v_i$ and $w_i = c$, this is just the previous matrix.

Tangential interpolation implies a solution

To summarize

Lemma

Suppose $\mathcal{L}$ is a tangential family for $\mathcal{A}$. If $f_1, \ldots, f_n \in \mathcal{A}$ satisfy

$$\sum_{i=1}^{n} M_{f_i}^L (M_{f_i}^L)^* \geq c^2 I_L, \; L \in \mathcal{L}$$

then there are $g_1, \ldots, g_n \in \mathcal{A}$ such that

$$\sum_{i=1}^{n} f_i g_i = 1 \text{ and } \|[M_{g_1}, \ldots, M_{g_n}]^T\| \leq c^{-1}$$

When does an algebra of multipliers $\mathcal{A}$ admit a tangential family?

Definition

A weak*-closed subspace $\mathcal{S}$ of $B(H)$ is said to be **elementary** if every $\varphi \in S_*$ with $\|\varphi\| < 1$ can be factored as

$$\varphi(A) = \langle Ax, y \rangle, \; A \in \mathcal{S}$$

for some $x, y \in H$ with $\|x\|\|y\| < 1$.

Define the column space of $\mathcal{A}$:
$C(\mathcal{A}) := \{[M_{g_1}, \ldots, M_{g_n}]^T : g_i \in \mathcal{A}\} \subset B(H^{(n)}, H)$

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Theorem

Suppose $C(\mathcal{A})$ is elementary. Then $\{\mathcal{A}[h] : h \in H\}$ is a tangential family for $\mathcal{A}$.

Proof.

- Let $\mathcal{J} = \{G \in C(\mathcal{A}) : \langle G(\lambda_i), v_i \rangle_{\mathbb{C}^n} = 0 \text{ for } i = 1, \ldots, k\}$.

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- If $H \in C(\mathcal{A})$ is *any* column satisfying $\langle H(x_i), v_i \rangle_{\mathbb{C}^n} = w_i$, then $H + G$ is also a solution for any $G \in \mathcal{J}$.

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- We have a contractive solution if and only if $\text{dist}(H, \mathfrak{J}) \leq 1$.

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- If $H \in C(\mathcal{A})$ is *any* column satisfying $\langle H(x_i), v_i \rangle_{\mathbb{C}^n} = w_i$, then $H + G$ is also a solution for any $G \in \mathcal{J}$.
- We have a contractive solution if and only if $\operatorname{dist}(H, \mathfrak{J}) \leq 1$.
- A standard distance argument shows that

$$\left[\left(\langle v_i, v_j \rangle - w_i \overline{w_j} \right) \langle k^L_{\lambda_i}, k^L_{\lambda_j} \rangle \right]_{i,j=1}^{k} \geq 0, \; L \in \mathcal{L}$$

implies $\operatorname{dist}(H, \mathfrak{J}) \leq 1$ when $C(\mathcal{A})$ is elementary.

∎

Main Result

We say that a function $h \in H^2_d$ is **outer** if $\mathcal{M}(H^2_d)[h] = H^2_d$.

Theorem

Suppose $\mathcal{A} \subset \mathcal{M}(H^2_d)$ is a unital, weak*-closed subalgebra. Then $C(\mathcal{A})$ is elementary and every $\varphi \in C(\mathcal{A})_*$ can be factored as

$$\varphi(A) = \langle Ah, k\rangle$$

where h is an outer function.
In other words $\mathcal{L} := \{\mathcal{A}[h] : h \text{ outer}\}$ is a tangential family for $\mathcal{A}$.

Main result

Corollary

Suppose $\mathcal{A}$ is a unital, weak*-closed subalgebra of $\mathcal{M}(H^2_d)$ and $f_1, \ldots, f_n \in \mathcal{A}$ satisfy

$$\sum_{i=1}^{n} M^L_{f_i} (M^L_{f_i})^* \geq c^2 I_L$$

for every L of the form $\mathcal{A}[h]$ where h is an outer function. Then there are $g_1, \ldots, g_n \in \mathcal{A}$ so that

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Corollary

Suppose $\mathcal{A}$ is a unital, weak-closed subalgebra of $\mathcal{M}(H^2_d)$ and $f_1, \ldots, f_n \in \mathcal{A}$ satisfy*

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$$\sum_{i=1}^{n} f_i g_i = 1 \text{ and } \|[M_{g_1}, \ldots, M_{g_n}]^T\| \leq c^{-1}$$

When $\mathcal{A} = \mathcal{M}(H^2_d)$, this is the Ball-Trent-Vinnikov result. For $d = 1$ it is the Raghupathi-Wick result.

Additional examples

For $\Omega \subset \mathbb{C}^d$ recall the Bergman space $L^2_a(\Omega)$ and its multipliers $M(L^2_a(\Omega)) = H^\infty(\Omega)$.

Theorem (Bercovici 1987)

$C(H^\infty(\Omega))$ is an elementary subspace of $B(L^2_a(\Omega), L^2_a(\Omega) \otimes \ell^2_n)$.

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For $h \in L^2_a(\Omega)$, the cyclic subspace generated by h may be identified with $L^2_a(\Omega, |h|^2 d\mu)$.
A sufficient condition to solve the Toeplitz corona problem for these algebras is

$$M_F^\nu (M_F^*)^\nu \geq c^2 I_\nu$$

for the absolutely continuous measures $\nu = |h|^2 \mu$ on Ω.