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Some Functional Analytic Properties of the Banach Algebras $A_p^r(G) = A_p(G) \cap L^r(G)$.

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1. The RNP.

A Banach space X has the **RNP** iff its
unit ball wants to be weakly compact, but just cannot make it,
as beautifully put by Jerry Uhl.

In fact X has the **RNP**, iff
any norm bounded closed convex subset Y is the closed convex hull of its strongly exposed points.

For ***dual Banach spaces X*** *this is equivalent to*
any such Y being the norm closed convex hull of its extreme points i.e. X having the Krein-Milman property (KMP)
(see [DiU] p.128, for at least 17 conditions which are equivalent to the **RNP**).

If X is isomorphic to $\ell^1(\Gamma)$ then X has the **RNP**.

The Fourier Algebra of the torus $\mathbb{T}$, namely A($\mathbb{T}$), is in fact $\ell^1(\mathbb{Z})$ and as such has the Radon-Nikodym property (**RNP**).

However $L^1(\mathbb{R}), hence\ A(\mathbb{R})$, *does not* possess this property.

Clearly A(G) *has the **RNP** if G is compact abelian* and yet, A(G) *does not have the **RNP** if G is abelian but non compact*.

However if K is any compact subset of the abelian group G then $A_K(G) = \{u \in A(G); sptu \subset K\}$ *does have the **RNP*** (here $sptu = cl\{x \in G; u(x) \neq 0\}$ and *cl* denotes closure).

The above results can be proved using the Fourier transform and other tools of *abelian* harmonic analysis. *These are not available anymore if G is not abelian*.

For any G, let $\forall 1 < p < \infty, A_p(G)$ denote the Figa-Talamanca-Herz Banach algebra as defined by Herz in [Hz1], thus generated by $L^{p'} * (L^p)^\vee$. Hence $u \in A_p(G), iff\ u = \sum u_n * v_n$,where

$u_n \in L^{p'}(G), v_n \in L^p(G), \sum \|u\|_{L^{p'}} \|v_n\|_{L^p} < \infty$, the infimum of these

being the norm of $u \in A_p$, and $p^{-1} + (p')^{-1} = 1$, $1 < p < \infty$.

Hence $A_2(G)$ is the Fourier algebra of G, (i.e. A(G) a la Eymard in [Ey1]).

If G is abelian then $A_2(G) = L^1(\hat{G})^\wedge$.

Remarks: If $p \neq 2$ then $A_p(G)$ is very different from $A_2(G)$. Since, if G_1, G_2 are compact abelian groups and $A_p(G_1)^*, A_p(G_2)^*$ are isometric as Banach spaces then G_1, G_2 are isomorphic as topological groups, as proved by Benyamini and Lin [BL]
While $A_2(G)^*$ is isometric to $\ell^\infty(\mathbb{Z})$, for all infinite metric compact abelian

It is the purpose of this paper to investigate the **RNP** for the Figa-Talamanca-Herz-Lebesgue Banach Algebras $A_p^r(G) = A_p \cap L^r(G), \forall 1 \le r \le \infty$, equipped with the norm $\|u\|_{A_p^r} = \|u\|_{A_p} + \|u\|_{L^r}$.

For *abelian G*, the Banach algebras $A_2^r(G) = \left\{ f \in L^1\left(\hat{G}\right); \hat{f} \in L^r(G) \right\}$ have been around for a long time. Their study started in a beautiful paper of Larsen, Liu and Wang [LLW1964]. The first important paper in which $A_p^1(G)$, for *arbitrary* G, was studied is by Hang-Chin Lai and Ing-Sheun Chen [LaCh1981]. It was this paper which gave us the impetus to study in [Gr2006] functional analytic properties of the Banach algebras $A_p^r(G)$, for arbitrary G.

We have proved in [Gr2] that for *any* G and any compact $K \subset G$ and any $1 < p < \infty$, $A_K^p = \{u \in A_p(G); sptu \subset K\}$ *has the **RNP.*** It has been proved by W. Braun, in an unpublished manuscript [Br1985?], that $A_p^1(G)$ has the **RNP**, if G is amenable, *employing the method used in our paper [Gr2].*

The main result of this paper is the

Theorem 5: Let G be unimodular and either (1) G be amenable and $1 < p < \infty$, or (2) p=2 and $A_2(G)$ have a multiplier bounded approximate identity . Then

(i) The Banach algebras $A_p^r(G), 1 \le r \le \max(p, p')$ have the RNP.

(ii) ***If G is SL(2,R) or SL(2,C) then $\forall\, 2 < r \le \infty, A_2^r(G)$ DOES NOT HAVE THE RNP.***

A crucial step in the proof is the identification of $A_p^r(G)\ \forall 1 \le r \le p$ as a dual Banach space, namely

Theorem 2: Let $1 < p < \infty$ and G be amenable, or p=2 and G be arbitrary. Then

(i) $A_p^r(G) = W_p \cap L^r(G)$, $\forall 1 \leq r \leq p$, is a dual Banach space.

(ii) The above equality fails even if G=Z, p=2 and r>2.

Here $W_p(G) = PF_p(G)^*$, where $PF_p(G)$ is the norm closure of $L^1(G)$ *in* $PM_p(G) = A_p(G)^*$.

2. Weak sequential completeness. (w.s.c)

Theorem: ***(a) Assume that for some $1 < q < \infty$, $A_q(G)$ is w.s.c.***

Then $A_q^r(G)$ **is w.s.c.** $\forall 1 \leq r < \infty$.

(b) $A_2^r(G)$ **is w.s.c.** $\forall 1 \leq r \leq \infty$. ***[Gr]***

Question: For which p and G is $A_p(G)$ w.s.c. ?

3. The strict containment Theorem.

If $r < s \text{ then } A_p^r(G) \subset A_p^s(G)$.

When is this a strict inequality?

Theorem*: Let G be a noncompact l.c. group and* $1 < p < \infty$.

(a) If $1 \le r < p \text{ and } r < s \le \infty, \Rightarrow A_p^r(G) \subsetneq A_p^s(G)$.

(b)* If G *is unimodular and

$1 \le r < \max(p, p') \text{ and } r < s \le \infty$, $\Rightarrow A_p^r(G) \subsetneq A_p^s(G)$.

(c) If G is amenable

and $1 \le r < s \le \infty, \Rightarrow A_p^r(G) \subsetneq A_p^s(G)$. ***[Gr]***

Remark: *Both* (a) and (b) cannot be improved much. Since, if G is a connected semisimple Lie group with finite centre (hence G is unimodular) then $A_2^r(G) = A_2(G)\ \forall r > 2$, by M.Cowling impressive result [Co]. In this case p=p'=2 and

For $2 < r < s < \infty$, $A_2(G) = A_2^r(G) = A_2^s(G)$.

Moreover, if $2 = r < s \text{ then } A_2^2 \subsetneq A_2^s = A_2$,

since for any G, if $A_2^2 = A_2$ then G is compact, (Rickert).

4. Nonfactorisation.

In improving a result of Burnham [Bu2] , Lai and Chen [LaCh1981] thm.3.3 have proved that for any noncompact locally compact group G the algebra $A_p^1(G)$ does not factorise.

We improve this result to the algebras $A_p^r(G), \forall 1 \leq r < \infty$. Lai and Chen's result is assumed in the proof for r=1, and use is made of the above strict containment theorem.

Theorem: For any noncompact locally compact group G and any $1 \le r < \infty$, the algebra $A_p^r(G)$ does not factorise.

Proof: Assume at first that $1 < r < \infty$. If $A_p^r \cdot A_p^r = A_p^r$, let $u \in A_p^r$. Then for any n ,there exist $u_1,...,u_n \ in \ A_p^r$ such that $u = u_1..u_n$, where $u_i \in A_p \cap L^r$.

By Burnham's lemma A in [Bu2] , $u \in L_1$. It follows that $A_p^r(G) = A_p^1(G)$. Since r > 1 , it follows from our Strong Containment Theorem in [Gr1] that this cannot be. The Lai-Chen result for r=1 completes the proof. **QED**

5. Arens Regularity

Theorem: ***(a) If G is a nondiscrete locally compact group then***
$A_p^r(G)$ ***is NOT Arens Regular for any***
$1 < p < \infty, 1 \leq r \leq \infty$.
(b) If G is any discrete group and
$1 < p < \infty \; and \; 1 \leq r \leq \max(p, p')$,
then $A_p^r(G)$ ***IS Arens Regular.***

(c) If G is discrete and $\max(p, p') < r < \infty$, *nothing on Arens Regularity is known, even if G is abelian , in fact even if* $G = \mathbb{Z}$ *and p=2.*
(d) If G is infinite, ***discrete and amenable*** *[**Abelian**] then* $A_2(G)$, $[A_p(G)]$ *IS NOT Arens Regular ,(Tony Lau and J.S.C. Wong, [B. Forrest]) respectively.*

6. Spectral Synthesis.

Theorem: *Assume that* $A_p(G)$ *has an approximate identity which is bounded in multiplier norm. (Abelian or amenable groups have such)*

Then for fixed $1 < p < \infty$, *the algebras* $A_p^r(G), \forall 1 \le r \le \infty$, *have the same sets of synthesis.*

This improves somewhat a result in [RS] Ch.6.

It has been proved by P. Eymard that If 1<p<3/2 [3/2<p<2, or p=2] then the unit sphere S^3, *in* R^4

IS [IS NOT] *a synthesis set for* $A_p(R^4)$.

Theorem: If 1<p<3/2, [3/2<p<2, or p=2] then

S^3 ***IS [IS NOT] a***

Synthesis set for $A_p^r(R^4), \forall 1 \le r \le \infty$ ∎

References.

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