A new (weak* limit) proof of spectral synthesis for singletons

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abstract

That singletons obey spectral synthesis has been known for more than 75 years. (That is, if $f \in A(G)$ has $f(x) = 0$, then there exists $f_n \in A(G)$ with $f_n = 0$ in a neighbourhood of x and $\|f - f_n\| \to 0$). A new proof of this result is given, using a striking lemma of Varopoulos. The lemma, which deserves to be better known, raises an interesting question.

From joint work with Kathryn Hare
(*Interpolation and Sidon sets for compact groups*, C.M.S.Books in Mathematics)

Outline

Notation

$A(X)$: restriction of the Fourier algebra to X
$\|f\|_{A(X)}$ the norm
$PM(X)$: the dual space of $A(X)$

$B(\mathbf{F})$: restriction of the Fourier-Stieltjes algebra to $\mathbf{F}$
$\|f\|_{B(\mathbf{F}}$ the norm

G, Γ: compact abelian group; its dual group;
$e, \mathbf{1}$ their respective identities
$\bar{\Gamma}$: the Bohr compactification of Γ
G_d: the discrete version of G

e: the identity of G

$\mathbf{1}$: the identity of Γ

A speculation - with the slimmest of evidence

Speculation Let $F \subset G$ and $\mathbf{E} \subset \Gamma$, both finite. Then there exists C depending only on card $\mathbf{E}$ such that

$$\|\hat{\mu}\|_{A(F)} \leq C \|\hat{\mu}\|_{\ell^\infty(F)} \text{ for all } \mu \in M(\mathbf{E}).$$

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$g \equiv 0$ in a neighbourhood of X.

Two lemmata

It is not hard to show that spectral synthesis for singletons follows from

Lemma 1. Let $\mathbf{H} \subset \Gamma$ be finite with $\mathbf{1} \in \mathbf{H}$ and $\varepsilon > 0$. Then there exists a compact e-neighbourhood $U \subset G$ such that, for all $\gamma \in \mathbf{H}$,

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The key step in the proof of Lemma 1 is:

Lemma 2.[Varopoulos 1965/Bourgain 1987] For every finite $F \subset G$ and $\gamma \in \Gamma$,

$$\left\|\widehat{\delta}_{\mathbf{1}} - \widehat{\delta}_{\gamma}\right\|_{A(F)} \leq (\pi/2)\left\|\widehat{\delta}_{\mathbf{1}} - \widehat{\delta}_{\gamma}\right\|_{\ell^{\infty}(F)}. \tag{2}$$

Proof of Lemma 1 using Lemma 2, I

We let $\varepsilon > 0$. Choose an e-neighbourhood U_1 such that

$$|\widehat{\delta}_{\mathbf{1}} - \widehat{\delta}_\gamma(x)| < \varepsilon/\pi \text{ for } x \in U_1 \text{ and } \gamma \in \mathbf{H}.$$

We shall find an e-neighbourhood $U \subset U_1$ such that

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holds.

By Lemma 2, $\|\mathbf{1} - \widehat{\delta}_\gamma\|_{A(F)} < \varepsilon/2$ for all finite sets $F \subset U_1$. For each such F, let $\mu_F \in \ell^1(\Gamma) = M(\Gamma)$ be such that $\widehat{\mu}_F = \mathbf{1} - \widehat{\delta}_\gamma$ on F and $\|\mu_F\| < \varepsilon/2$.

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$\hat{\mu}_\infty$ may not be continuous (or even measurable).

Proof of Lemma 1 using Lemma 2, II

Repeating,

$$\|\mu_\infty\|_{M(\bar{\Gamma})} = \|\widehat{\mu}\|_{B(G_d)} \leq \varepsilon/2 \ \text{ and } \quad \widehat{\mu}_\infty = 1 - \widehat{\delta}_\gamma \text{ on } U_1.$$

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Since $h\widehat{\mu}_\infty$ is continuous on G, $h\widehat{\mu}_\infty \in B(G) = A(G)$. Hence, $\|1 - \widehat{\delta}_\gamma\|_{A(U)} = \|h\widehat{\mu}_\infty\|_{A(U)} < \varepsilon$. □

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Lemma 2 (again) For every finite $F \subset G$ and $\gamma \in \Gamma$,

$$\|\widehat{\delta}_{\mathbf{1}} - \widehat{\delta}_\gamma\|_{A(F)} \leq 3\pi/4 \|\widehat{\delta}_{\mathbf{1}} - \widehat{\delta}_\gamma\|_{\ell^\infty(F)}.$$

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$$\begin{aligned} \|\hat{\delta}_{\mathbf{1}} - \delta_\gamma\|_{A(F)} &= \sup\{|\hat{\nu}(\mathbf{1}) - \hat{\nu}(\gamma)| : \nu \in M(F), |\hat{\nu}\|_\infty \le 1\} \\ &= \sup\{|f(1) - f(\gamma)| : f \in C_F(\Gamma), \|f\|_\infty \le 1\|. \end{aligned}$$

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Therefore $\|1 - e^{ix}\|_{A([-\tau,\tau])} \leq \tau$.

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B.) For $0 < \varepsilon$,

$$1 - e^{i\theta} = \sum_{k \in \mathbb{Z}} c_k e^{ik\theta} \text{ if } |\theta| \leq h \leq \pi/2, \text{ where } \sum_{k \in \mathbb{Z}} |c_k| \leq (1 + \varepsilon)h. \tag{3}$$

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Suppose $\gamma \in \Gamma$ satisfies $\sup_{x \in F} |1 - \gamma(x)| = 2\sin(\tau/2)$, where $0 < \tau \leq \pi$. Then for each $x \in F$ we may write $\gamma(x) = e^{i\theta}$, where $|\theta| \leq \pi$.

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By (3) we have

$$1 - \gamma(x) = \sum_{k \in \mathbb{Z}} c_k \gamma(x^k). \tag{4}$$

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Let $f \in C_F(\Gamma)$. Then by (4),

$$f(\mathbf{1}) - f(\gamma) = \sum_{x \in F} \hat{f}(x)(1 - \gamma(x)) = \sum_{k \in \mathbb{Z}} c_k (\sum_{x \in F} \hat{f}(x)\gamma(x^k))$$
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Varopoulos (1965), not Bourgain (1987), not Rodríguez-Piazza

Lust(-Piquard) 1987

Thank you.