Simplicial and Cyclic Cohomology of Banach algebras

Frédéric Gourdeau

Université Laval

BA2011, August 4

Some motivation within Hochschild cohomology for Banach algebras

- Amenability : $H^n(A, X') = 0$ all $n \geq 1$, all dual-module X'.
- $L^1(G)$ amenable iff G amenable (B. E. Johnson)
- Weaker notion : weak-amenability $H^1(A, A') = 0$
- Simplicially trivial $H^n(A, A') = 0$ all $n \geq 1$: between amenable and weak-amenable

Group and semigroup ℓ^1 algebras

- Topological group G or topological semigroup S - we will consider discrete semigroups only in this talk.
- Product given by convolution: abuse of notation $a \in \ell^1(S)$ written as
$$a = \sum_{s \in S} a_s \delta_s = \sum_{s \in S} a_s s.$$
- Product given by $(ab)_s = \sum_{s=tu} a_t b_u$.

- Initial work. Bowling, Dales, Duncan: some specific low dimension cases
- $\ell^1(\mathbb{N})$ for $(\mathbb{N}, +)$ (GJW, TAMS 2005) - includes more general semigroups : simplicial cohomology vanish for $n \geq 2$
- $\ell^1(\mathbb{N}^k)$ for $(\mathbb{N}, +)$ (GLW, SM 2005) : vanish for $n \geq k+1$, explicit description through Kunneth formulae for lower degrees
- Similar for $L^1(\mathbb{R}_+^k)$ for $(\mathbb{R}_+, +)$ (GLW - Proceedings Edmonton 2004)
- $\ell^1(\mathbb{N})$ for $(\mathbb{N}, \max)$: simplicially trivial
- $L^1(\mathbb{R}_+)$ for $(\mathbb{R}_+, max)$ (Elliott – currently PhD project with M. C. White)

- Semilattice (Choi, Glasgow MJ 2006, Houston JM 2010) - so includes $(\mathbb{N}, max)$: simplicially trivial
- Rees semigroup (GGW, 2011) : get the cohomology of the underlying group
- Bicyclic semigroup (GW, Quart. Oxf. J., 2011) : simplicial cohomology vanishes for $n \geq 2$
- Band semigroup (CGW, 2012?): simplicial trivial
- Cuntz semigroup (GW, submitted): simplicial cohomology vanishes for $n \geq 2$

- Semilattice (Choi, Glasgow MJ 2006, Houston JM 2010) - so includes $(\mathbb{N}, max)$: simplicially trivial
- Rees semigroup (GGW, 2011) : get the cohomology of the underlying group
- Bicyclic semigroup (GW, Quart. Oxf. J., 2011) : simplicial cohomology vanishes for $n \geq 2$
- Band semigroup (CGW, 2012?): simplicial trivial
- Cuntz semigroup (GW, submitted): simplicial cohomology vanishes for $n \geq 2$

Simplicial triviality - no general amenability type property (yet ?).

- A a Banach algebra, $Y = A'$ as an A-bimodule ($a \cdot y$ and $y \cdot a$)
- A n-cochain T : bounded n-linear from A^n to A', or linear on $n+1$-fold tensor product.

Then $\delta^n : \mathcal{C}^n(A, Y) \to \mathcal{C}^{n+1}(A, Y)$ is for $\mathbf{x} = a_1 \otimes \cdots \otimes a_{n+1}$

$$\begin{aligned}(\delta^n T)(\mathbf{x}) &= T(a_2 \otimes \cdots \otimes a_{n+1} \cdot a_1) \\ &\quad + \sum_{j=1}^{n} (-1)^j T(a_1 \otimes \cdots \otimes a_j \cdot a_{j+1} \otimes \cdots \otimes a_{n+1}) \\ &\quad + (-1)^{n+1} T(a_1 \otimes \cdots \otimes a_n \cdot a_{n+1})\end{aligned}$$

where $a_1, \ldots a_n \in A$.

- T *n-cocycle* if $\delta^n T = 0$
- T *n-coboundary* if $T = \delta^{n-1} S$ for some $S \in \mathcal{C}^{n-1}(A, Y)$.

Question : is $\mathcal{H}^n(A, A') = 0$ i.e is a cocycle necessarily a coboundary?

Cyclic cohomology

The *n*-cochain T is *cyclic* if

$$T(a_1 \otimes \cdots \otimes a_{n+1}) = (-1)^n T(a_{n+1} \otimes a_1 \otimes \cdots \otimes a_n)$$

Cyclic *n*-cochains: $\mathcal{CC}^n(A)$.
Cyclic cochains $\mathcal{CC}^n(A)$ form a subcomplex of $\mathcal{C}^n(A, A')$, that is

$$\delta^n : \mathcal{CC}^n(A) \to \mathcal{CC}^{n+1}(A).$$

So one defines $\mathcal{HC}^n(A)$.
The cyclic and simplicial cohomology groups are connected via the Connes-Tzygan long exact sequences.

Connes-Tzygan

If A is H-unital, the Connes-Tzygan long exact sequence exists

$$0 \to \mathcal{HH}^1(A) \to \mathcal{HC}^0(A) \to \mathcal{HC}^2(A) \to \mathcal{HH}^2(A) \to \mathcal{HC}^1(A) \to \cdots$$
$$\to \mathcal{HH}^j(A) \to \mathcal{HC}^{j-1}(A) \to \mathcal{HC}^{j+1}(A) \to \mathcal{HH}^{j+1}(A) \to \mathcal{HC}^j(A) \to \cdots$$

Simplicial vanishes then:

$$0 \to \mathcal{HC}^{j-1}(A) \to \mathcal{HC}^{j+1}(A) \to 0$$

Cyclic "nearly" vanishes (i.e. constant for odd and constant for even):

$$\mathcal{HC}^{j-1}(A) \to \mathcal{HC}^{j+1}(A) \to \mathcal{HH}^{j+1}(A) \to \mathcal{HC}^j(A) \to \mathcal{HC}^{j+2}$$

Building a contracting homotopy in cyclic cohomology

For cohomology, we have

$$\mathcal{C}^{n+1}(A, A') \xleftarrow{\delta^n} \mathcal{C}^n(A, A') \xleftarrow{\delta^{n-1}} \mathcal{C}^{n-1}(A, A')$$

This is the dual of the homology where we look at

$$\widehat{\bigotimes}^{n+2} A \xrightarrow{d^n} \widehat{\bigotimes}^{n+1} A \xrightarrow{d^{n-1}} \widehat{\bigotimes}^{n} A$$

Building a contracting homotopy in cyclic cohomology

$$\widehat{\bigotimes}^{n+2} A \xrightarrow{d^n} \widehat{\bigotimes}^{n+1} A \xrightarrow{d^{n-1}} \widehat{\bigotimes}^{n} A$$

If we can construct maps $s^k : \widehat{\bigotimes}^{k+1} A \to \widehat{\bigotimes}^{k+2} A$ for $k = n-1, n$ such that $d^n s^n + s^{n-1} d^{n-1} = Id$ (the identity map) then we have $\mathcal{H}^n(A, A') = 0$. We do not obtain this directly in general.
Rather we build s^n such that its dual σ^n acts in cyclic cohomology and is such that (essentially) $\sigma^n \delta^n + \delta^{n-1} \sigma^{n-1} = I$ in cyclic cohomology.
Why should that be an easier approach?

How is s^n constructed?

For $\rho : A \to A\hat{\otimes}A$ we define $s_i^n : \mathcal{C}_n(A, A) \to \mathcal{C}_{n+1}(A, A)$, $1 \leq i \leq n+1$, by

$$s_i^n(x_1 \otimes \cdots \otimes x_{n+1}) = (-1)^i (x_1 \otimes \cdots \otimes x_{i-1} \otimes \rho(x_i) \otimes x_{i+1} \otimes \cdots \otimes x_{n+1}) .$$

Then $s^n = \sum_{k=1}^{n+1} s_k^n$.

How is s^n constructed?

For $\rho : A \to A\hat{\otimes}A$ we define $s_i^n : \mathcal{C}_n(A, A) \to \mathcal{C}_{n+1}(A, A)$, $1 \leq i \leq n+1$, by

$$s_i^n(x_1 \otimes \cdots \otimes x_{n+1}) = (-1)^i (x_1 \otimes \cdots \otimes x_{i-1} \otimes \rho(x_i) \otimes x_{i+1} \otimes \cdots \otimes x_{n+1}) \,.$$

Then $s^n = \sum_{k=1}^{n+1} s_k^n$.
In **cyclic cohomology**, when considering the adjoint of $d^n s^n + s^{n-1} d^{n-1}$, most terms cancel for any ρ and those left involve expressions like

$$\rho(ab) + \pi\rho(a) \otimes b - a\rho(b) - \rho(a)b.$$

We need the identity to come out of this, and as little else as possible! Typically we aim for $\pi\rho(a) = a$ and

$$\rho(ab) - a\rho(b) - \rho(a)b = 0$$

or "simpler" than the initial tensor.

To be more precise, we get sums of terms of the type

$$
\begin{aligned}
&+ \;\; x_1 \otimes \cdots \otimes x_{i-1} \otimes \rho(x_i x_{i+1}) \otimes x_{i+2} \otimes \cdots \otimes x_{n+1} && (1)\\
&+ \;\; x_1 \otimes \cdots \otimes x_{i-1} \otimes \pi\rho(x_i) \otimes x_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_{n+1} && (2)\\
&- \;\; x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i\rho(x_{i+1}) \otimes x_{i+2} \otimes \cdots \otimes x_{n+1} && (3)\\
&- \;\; x_1 \otimes \cdots \otimes x_{i-1} \otimes \rho(x_i)x_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_{n+1} && (4)
\end{aligned}
$$

$\mathbf{S}_m$ $(1 \leq m \leq \infty)$: abstract semigroup generated by a unit $\mathbf{1}$, a zero z_0 and p_i, q_i, $1 \leq i \leq m$ such that

$$q_i p_j = \delta_{ij}\mathbf{1}.$$

Apart from $\mathbf{1}$ and z_0, elements are $p_\alpha q_\beta$ where α, β are finite strings of integers.
Product

$$(p_\alpha q_{\beta\gamma})(p_\beta q_{\beta'}) = p_\alpha q_{\beta'\gamma}$$

and

$$(p_\alpha q_\beta)(p_{\beta\gamma} q_{\beta'}) = p_{\alpha\gamma} q_{\beta'},$$

where $\alpha\beta$ denotes the string formed by the integers in α followed by those in β. Note that $q_\beta p_\beta = \mathbf{1}$, $p_\alpha p_\beta = p_{\alpha\beta}$ and $q_\alpha q_\beta = q_{\beta\alpha}$.

This semigroup is a path semigroup: take directed graph of m directed loops at a single vertex.

- α is a directed path, simply the path formed by walking along the loops in the order they appear in α.
- p_α means to go forward along this path
- q_α to go backwards along this same path
- The semigroup contains paths which consist of going forward along a path, and then going backward along a path (the same or different).
- The rules for the product are that if you go backward on a loop and then forward on a loop, the product is zero unless it is the same loop, in which case the product is $\mathbf{1}$. **Hence the length of a product of two paths is not the sum of the lengths: cancelation can occur.**

Cuntz semigroup: a two step process.
Step 1 : The idea is to use $\rho(p_\alpha q_\beta) = p_\alpha \otimes q_\beta$ and iterate. This moves us to a world where there is no cancelation.

Step 2 : The idea is to take something close to

$$\rho(p_\alpha q_\beta) = \sum_{k=0}^{N-1} p_{\alpha_{[1,k]}} \otimes p_{\alpha_{[k+1,n]}} q_\beta + \sum_{l=1}^{M} p_\alpha q_{\beta_{[l+1,m]}} \otimes q_{\beta_{[1,l]}},$$

where the term corresponding to $k=0$ is $\mathbf{1} \otimes p_\alpha q_\beta$, and the term for $l=M$ is $p_\alpha \otimes q_\beta$.

Notes

1- ρ is and unbounded derivation and we need to take some averages.

2- This is possible because of step 1.

3- This map is analogous to the map used for $\ell^1(Z_+)$.

4- A map entirely analogous to this works for $\ell^1(F)$ for F the free semigroup on a finite or infinite number of symbols.

Theorem

The cyclic cohomology of the Cuntz algebra $\ell^1(\mathbf{S}_m)$ is zero in odd dimensions, and is $\mathbb{C}$ in even dimensions greater than 0.

Theorem

The simplicial cohomology of the Cuntz algebra $\ell^1(\mathbf{S}_m)$ is zero in degrees 2 and above, and the first simplicial cohomology group is isomorphic to the space of traces vanishing at $\mathbf{1}$.

Conclusion

Doing higher order cyclic cohomology is not so difficult and not so different from cyclic cohomology of degrees 1 and 2 when this approach works. The difficulty is not so much with the degree but lies with the Banach algebra you are considering.