Finitely Correlated θ-commtuing Row-Isometries

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Introduction

I will talk today on my paper:
Finitely Correlated Representations of Product Systems of C^*-Correspondences over $\mathbb{N}^k$, J. Funct. Anal. 260 (2011)

Row Operators

Let $\mathcal{H}$ be a Hilbert space. A row-operator is a bounded map from $\mathcal{H}^{(n)}$ to $\mathcal{H}$ of the form $A = [A_1, \ldots, A_n]$, where each $A_i \in \mathcal{B}(\mathcal{H})$.

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- and A is a *row-coisometry* when A is coisometric (i.e. when $\sum_{i=1}^n A_i A_i^* = I$).
- We call A a *row-unitary* when it is both isometric and coisometric.

θ-commuting Row-isometries

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The pair (A, B) can be seen as a representation of the unital semigroup

$$\mathbb{F}_\theta^+ = \langle e_1, \ldots, e_m, f_1, \ldots, f_n : e_i f_j = f_{j'} e_{i'} \text{ when } \theta(i,j) = (i',j') \rangle.$$

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We will be concerned with the case when S and T are finitely correlated row-unitaries.

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The representation (S, T) is a *minimal* dilation when $\mathcal{H}$ is cyclic for (S, T), i.e.

$$\mathcal{K} = \bigvee_{e_u f_w \in \mathbb{F}_\theta^+} S_u T_w \mathcal{H}.$$

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- Two θ-commuting row-isometries S and T are finitely correlated
- S and T are the minimal isometric dilation of two θ-commuting row-contractions A and B on a finite dimensional space.

Isometric Dilations Existence

Theorem (Solel (2006); Davidson, Power & Yang (2010))

If A and B are θ-commuting row-contractions on $\mathcal{H}$ then they have an isometric dilation S and T on $\mathcal{K} \supseteq \mathcal{H}$.
Further, if A and B are row-coisometries i.e.

$$\sum_{i=1}^{m} A_i A_i^* = I = \sum_{j=1}^{n} B_j B_j^*,$$

then if (S, T) is chosen to be minimal then S and T will be row-unitaries and will be unique (up to a unitary equivalence fixing $\mathcal{H}$).

Finitely Correlated Representations: Result

Theorem (F. 2011)

Let S, T be a θ-commuting finitely correlated row-unitaries on $\mathcal{K}$. Then there exists a unique minimal cyclic finite dimensional space $\mathcal{V} \subseteq \mathcal{K}$, i.e. there is a unique minimal $\mathcal{V}$ such that

$$\mathcal{K} = \bigvee_{e_u f_w \in \mathbb{F}_\theta^+} S_u T_w \mathcal{V}.$$

The compression of (S, T) to the space $\mathcal{V}$ is a complete unitary invariant for finitely correlated representations of $\mathbb{F}_\theta^+$.

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The compression of (S, T) to the space $\mathcal{V}$ is a complete unitary invariant for finitely correlated representations of $\mathbb{F}_\theta^+$. Moreover, if $\mathcal{S} = \overline{\mathrm{Alg}\{I, S_1, \ldots, S_m, T_1, \ldots, T_n\}}^{\mathrm{WOT}}$ then $P_\mathcal{V}$ is in $\mathcal{S}$ and $P_\mathcal{V} \mathcal{S} P_\mathcal{V}$ is a C^*-algebra.

Main Idea in the Proof

Theorem (F. 2011)

Let $A = [A_1, \ldots, A_m]$ and $B = [B_1, \ldots, B_n]$ be θ-commuting row coisometries. Let $S = [S_1, \ldots, S_m]$ and $T = [T_1, \ldots, T_n]$ be the minimal isometric dilation of A and B.

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Then the row-isometry $[S_1 T_1, S_1 T_2, \ldots, S_m T_n]$ is the minimal isomteric dilation of $[A_1 B_1, \ldots, A_m B_n]$.

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Then the row-isometry $[S_1 T_1, S_1 T_2, \ldots, S_m T_n]$ is the minimal isomteric dilation of $[A_1 B_1, \ldots, A_m B_n]$.

Corollary

Two θ-commuting row unitaries S and T are finitely correlated if and only if the single row-unitary $[S_1 T_1, S_1 T_2, \ldots, S_m T_n]$ is finitely correlated.

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This allows us to consider a single finitely correlated row-isometry, in place of two. Finitely correlated row-isometries were previously studied by Davidson, Kribs and Shpigel (2001).

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If E is complete with respect to the inner-product norm, then it is a C^*-correspondence.

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- If α is an automorphism on $\mathcal{A}$ we can form a C^*-correspondence ${}_\alpha\mathcal{A} = \mathcal{A}$ with the following left and right actions ($a \in \mathcal{A}$, $b \in {}_\alpha\mathcal{A}$)

$$a \cdot b = \alpha(a)b \text{ and } b \cdot a = ba$$

and inner-product: $\langle b, c\rangle = b^*c$.
Representations of this product system are covariant representations of the dynamical system $(\mathcal{A}, \alpha, \mathbb{N})$.

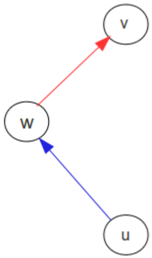
Product Systems of C^*-correspondences

Let $E_1, \ldots, E_k$ be C^*-correspondences over a common C^*-algebra $\mathcal{A}$. They form a product system over $\mathbb{N}^k$ if there are (associative) isomorphisms

$$t_{i,j} : E_i \otimes E_j \to E_j \otimes E_i.$$

Examples(1)

Higher-rank graphs can be described as product systems.
A Higher-Rank Graph algebra consists of a set of vertices V and k sets of directed edges E_k together with a commutation rule between the edges:
e.g. if you travel by a red edge followed by a blue edge, then you can equivalently travel by a blue edge followed by a red edge.



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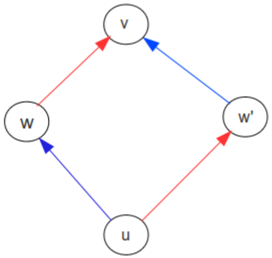
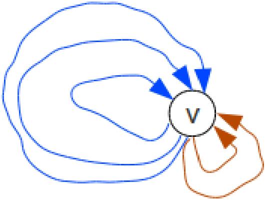


Examples(1)

Two row-contractive θ-commuting operators A and B describe a representation of a single vertex 2-graph.



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If $\alpha_1, \ldots, \alpha_k$ are k commuting automorphisms of $\mathcal{A}$ then the C^*-correspondences ${}_{\alpha_1}\mathcal{A}, \ldots, {}_{\alpha_k}\mathcal{A}$ form a product system.

Examples (2)

If $\alpha_1, \ldots, \alpha_k$ are k commuting automorphisms of $\mathcal{A}$ then the C^*-correspondences ${}_{\alpha_1}\mathcal{A}, \ldots, {}_{\alpha_k}\mathcal{A}$ form a product system. Representations of this product system are covariant representations of the dynamical system $(\mathcal{A}, \alpha, \mathbb{N}^k)$.

Main Result

Theorem (F. 2011)

Let S be a unitary finitely correlated representation of product system of C^*-correspondences over $\mathbb{N}^k$ on $\mathcal{K}$. Then there exists a unique minimal cyclic finite dimensional space $\mathcal{V} \subseteq \mathcal{K}$. The compression of S to the space $\mathcal{V}$ is a complete invariant for finitely correlated representations of the product system.

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Moreover, if $\mathcal{S} = Ran(\overline{S})^{\mathrm{WOT}}$ then $P_{\mathcal{V}} \mathcal{S} P_{\mathcal{V}}$ is a C^*-algebra.