Unbounded Tensor product Operator Algebras

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- This is a joint work with **A. Inoue** and **M. Weigt**

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I Introduction and preliminary definitions

It is well known that there are no bounded operators a, b such that

$$ab - ba = I, \text{ where } I \text{ is the identity operator.}$$

But in the physical applications there are unbounded operators that fulfil the preceding commutation relation. Take, for instance the Hilbert space $L^2(\mathbb{R})$, its dense subspace $S(\mathbb{R})$ (Schwarz space) and the operators

$$a = \frac{d}{dt}, \ b = T_{f_0} \text{ on } S(\mathbb{R}), \text{ with } f_0(t) := t, \ t \in \mathbb{R} \text{ and}$$
$$T_{f_0}(g) := f_0 g, \ g \in S(\mathbb{R}).$$

Then, $ab - ba = I$. Of course, there are several others examples, like for instance, the momentum and position operators, which fulfil the Heisenberg commutation relation. So, it is quite natural to consider algebras of unbounded operators.

Allan's GB^*-algebras (1967), generalizations of C^*-algebras are algebras of unbounded operators (Dixon, 1970); a typical example of a GB^*-algebra is Arens Algebra

$$L^\omega[0,1] = \bigcap_{1 \leq p < \infty} L^p[0,1]$$

equipped with the topology of the L^p-norms, $1 \leq p < \infty$. We have already considered tensor products of GB^*-algebras and among those is, for instance, the GB^*-algebra

$$C_c(X, L^\omega[0,1]) = C_c(X) \check{\otimes} L^\omega[0,1],$$

X a locally compact space, of all continuous $L^\omega[0,1]$-valued functions on X, with the compact open topology c. $L^\omega[0,1]$ can be replaced with any GB^*-algebra $\mathcal{A}$, whose "bounded part" coincides with the C^*-subalgebra $\mathcal{A}[B_0]$ through which the structure of $\mathcal{A}$ is studied.

In the present talk we shall present tensor products of GW^*-algebras, unbounded generalization of standard von Neumann algebras. GW^*-algebras were initiated by A. Inoue, in 1978, for studying the "Tomita-Takesaki" theory in algebras of unbounded operators. In 1971. P.G. Dixon considered the so-called EW^*-algebras (extended W^*-algebras), which are also algebras of unbounded operators that under a certain condition coincide with GW^*-algebras. In the study of either GB^*-algebras or GW^*-algebras, a crucial role is played by their "bounded part" (consisting of bounded operators), which in the first case is always a C^*-algebra and in the second case a von Neumann algebra.

In the definition of GW^*-algebras an essential role is played by the $\mathcal{O}^*$-algebras introduced by G. Lassner, in 1981, aiming to the solution of questions that appear in quantum statistics and quantum dynamics, that the algebraic formulation of quantum theories presented by Haag and Kastler (1964), could not face. A second important role is that of the unbounded commutants.

So, let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$ and

$$\mathcal{L}^\dagger(\mathcal{D}) := \{a : \mathcal{D} \to \mathcal{D} \text{ linear} : \mathcal{D}(a^*) \supset \mathcal{D} \text{ and } a^*\mathcal{D} \subset \mathcal{D}\},$$

where $\mathcal{D}(a^*)$ means domain of a^*. $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$-algebra under usual algebraic operations and involution $\dagger$ defined by $a^\dagger := a^* \upharpoonright \mathcal{D}$. A $*$-subalgebra $\mathcal{M}$ of $\mathcal{L}^\dagger(\mathcal{D})$ is called an $\mathcal{O}^*$-algebra on $\mathcal{D}$ in $\mathcal{H}$. $\mathcal{D}$ carries a locally convex topology called *graph topology*, induced by the elements of $\mathcal{M}$ and denoted by $\tau_\mathcal{M}$. If the locally convex space $\mathcal{D}[\tau_\mathcal{M}]$ is complete, then $\mathcal{M}$ is said to be a *closed $\mathcal{O}^*$-algebra*. If $\mathcal{H} := L^2(\mathbb{R})$ and $\mathcal{D} := C_0^\infty(\mathbb{R}) :=$ {smooth functions on $\mathbb{R}$ with compact support}, then the set of all differential operators

$$a := \sum_{n=0}^{k} f_n(t)(\frac{d}{dt})^n, \;\; f_n \in C^\infty(\mathbb{R}), k \in \mathbb{N} \cup \{0\},$$

acting on $\mathcal{D}$, is a closed $\mathcal{O}^*$-algebra on $\mathcal{D}$ in $\mathcal{H}$.

Now, let $\mathcal{M}$ be a closed $\mathcal{O}^*$-algebra on $\mathcal{D}$ in $\mathcal{H}$. Denote by $\mathcal{M}'_w$ the *weak commutant* of $\mathcal{M}$, where

$$\mathcal{M}'_w := \{a \in \mathcal{B}(\mathcal{H}) :< ax\xi|\eta >=< a\xi|x^\dagger\eta >\},$$

for all $x \in \mathcal{M}, \xi, \eta \in \mathcal{D}$. $\mathcal{M}'_w$ is a closed $*$-invariant subspace of $\mathcal{B}(\mathcal{H})$ with respect to the WOT, but it is not necessarily a von Neumann algebra, this happens when $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$. Another unbounded commutant of $\mathcal{M}$, which is an $\mathcal{O}^*$-algebra on $\mathcal{D}$ in $\mathcal{H}$, is given by

$$\mathcal{M}'_c := \{a \in \mathcal{L}^\dagger(\mathcal{D}) : ax = xa, \; \forall \; x \in \mathcal{M}\}.$$

Furthermore, an unbounded bicommutant of $\mathcal{M}$ is defined as follows: Suppose that $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$. Then,

$$\mathcal{M}''_{wc} := (\mathcal{M}'_w \upharpoonright \mathcal{D})'_c = \{a \in \mathcal{L}^\dagger(\mathcal{D}) : xa\xi = ax\xi, \\ \forall \; x \in \mathcal{M}'_w, \xi \in \mathcal{D}\}.$$

Definition. Let $\mathcal{M}$ be a a closed $\mathcal{O}^*$-algebra on $\mathcal{D}$ in $\mathcal{H}$. Then,

(1) (Inoue) $\mathcal{M}$ is called a *GW^*-algebra*, if $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$ and $\mathcal{M}''_{wc} = \mathcal{M}$.

(2) (Dixon) If $\mathcal{M}_b := \{x \in \mathcal{M} : \overline{x} \in \mathcal{B}(\mathcal{H})\}$ and $\overline{\mathcal{M}_b} := \{\overline{x} : x \in \mathcal{M}_b\}$, then $\mathcal{M}$ is said to be an *EW^*-algebra*, if it contains the identity operator, it is symmetric, in the sense that $(1 + x^*x)^{-1}$ exists and belongs to $\mathcal{M}_b$, for all $x \in \mathcal{M}$, and $\overline{\mathcal{M}_b}$ is a von Neumann algebra.

Note that *every EW^*-algebra is a GW^*-algebra*, see comments after the Proposition below. On the other hand, if the maximal $\mathcal{O}^*$–algebra $\mathcal{L}^\dagger(\mathcal{D})$ is closed (this, for instance, happens when $\mathcal{D}[\tau_{\mathcal{D}}]$ is complete), then $\mathcal{L}^\dagger(\mathcal{D})'_w = \mathbb{C}I$ and this implies that $\mathcal{L}^\dagger(\mathcal{D})$ is a *GW^*-algebra* with $(\mathcal{L}^\dagger(\mathcal{D})'_w)' = \mathcal{B}(\mathcal{H})$, but it is not an *$EW^*$-algebra*.

In general, when $\mathcal{M}$ is a GW^*-algebra containing the identity operator and

$$(\mathcal{M}'_w)'\mathcal{D} \subset \mathcal{D},$$

then $\mathcal{M}$ is an EW^*-algebra with $\overline{\mathcal{M}_b} = (\mathcal{M}'_w)'$ and the GW^*-algebra $\mathcal{M}''_{wc} = \mathcal{M}$ is the maximal EW^*-algebra with $\overline{\mathcal{M}_b} = (\mathcal{M}'_w)'$.

A useful characterization of GB^*-algebras is the following:

Proposition (Inoue) *Let $\mathcal{M}$ be a closed $\mathcal{O}^*$-algebra on $\mathcal{D}$ in $\mathcal{H}$, such that $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$. The following are equivalent*:

(1) *$\mathcal{M}$ is a GW^*-algebra*.

(2) $\mathcal{M} = \{a \in \mathcal{L}^\dagger(\mathcal{D}) : \overline{a}\eta(\mathcal{M}'_w)'\}$, *where* $\overline{a}\eta(\mathcal{M}'_w)'$ *means that the operator* $\overline{a}$ *is affiliated with the von Neumann algebra* $(\mathcal{M}'_w)'$.

Every EW^*-algebra fulfils the following properties:

$$\mathcal{M}_b\mathcal{D} \subset \mathcal{D},\ \overline{\mathcal{M}_b}'\mathcal{D} \subset \mathcal{D} \ \text{ and } \ \mathcal{M} = \{x \in \mathcal{L}^\dagger(\mathcal{D}) : \overline{x}\eta\overline{\mathcal{M}_b}\},$$

so that it is clear by all the above that every EW^*-algebra is a GW^*-algebra.

II Tensor products of GW^*-algebras

Let $\mathcal{D}_1, \mathcal{D}_2$ be dense subspaces of the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, respectively. Denote by $\mathcal{H}_1 \dot{\otimes} \mathcal{H}_2$ the *Hilbert space tensor product* of the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$. Then, the algebraic tensor product $\mathcal{D}_1 \otimes \mathcal{D}_2$ of $\mathcal{D}_1$ and $\mathcal{D}_2$ is a dense subspace of $\mathcal{H}_1 \dot{\otimes} \mathcal{H}_2$.

If $\mathcal{M}$ is an $\mathcal{O}^*$-algebra on $\mathcal{D}$ in $\mathcal{H}$, denote by $\widetilde{\mathcal{D}}$ the completion of $\mathcal{D}[\tau_{\mathcal{M}}]$. Put now $\widetilde{a} = \overline{a} \restriction \widetilde{\mathcal{D}}$, $a \in \mathcal{M}$ and $\widetilde{\mathcal{M}} := \{\widetilde{a} : a \in \mathcal{M}\}$. Then $\widetilde{\mathcal{M}}$ is the smallest closed extension of $\mathcal{M}$, which is called the *closure* of $\mathcal{M}$. It is easily shown that $\mathcal{M}$ is closed if and only if $\mathcal{M} = \widetilde{\mathcal{M}}$ if and only if $\mathcal{D} = \widetilde{\mathcal{D}} = \bigcap_{a \in \mathcal{M}} \mathcal{D}(\overline{a})$.

Suppose now that $\mathcal{M}_1, \mathcal{M}_2$ are $\mathcal{O}^*$-algebras on $\mathcal{D}_1, \mathcal{D}_2$ in $\mathcal{H}_1, \mathcal{H}_2$, respectively. Then the algebraic tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$ of $\mathcal{M}_1, \mathcal{M}_2$ is an $\mathcal{O}^*$- algebra on $\mathcal{D}_1 \otimes \mathcal{D}_2$ in $\mathcal{H}_1 \dot{\otimes} \mathcal{H}_2$. The closure of $\mathcal{M}_1 \otimes \mathcal{M}_2$ denoted by $\mathcal{M}_1 \widetilde{\otimes} \mathcal{M}_2$ is called *tensor product of the* $\mathcal{O}^*$*-algebras* $\mathcal{M}_1$ and $\mathcal{M}_2$.

The domain of the closed $\mathcal{O}^*$-algebra $\mathcal{M}_1\widetilde{\otimes}\mathcal{M}_2$ is denoted by $\mathcal{D}_1\widetilde{\otimes}\mathcal{D}_2$. For $a_1 \in \mathcal{M}_1$ and $a_2 \in \mathcal{M}_2$, $\overline{a_1 \otimes a_2} \upharpoonright \mathcal{D}_1\widetilde{\otimes}\mathcal{D}_2$ is denoted by $a_1\widetilde{\otimes}a_2$, and

$$\mathcal{M}_1\widetilde{\otimes}\mathcal{M}_2 := \{\sum_{k=1}^{n} a_{1,k}\widetilde{\otimes}a_{2,k},\ a_{1,k} \in \mathcal{M}_1,\ a_{2,k} \in \mathcal{M}_2\}.$$

The *W^*-tensor product* of two von Neumann algebras $\mathcal{M}_1, \mathcal{M}_2$ will be denoted by $\mathcal{M}_1\overline{\otimes}\mathcal{M}_2$.

Let's now see, when the tensor product of two $\mathcal{O}^*$-algebras results into a GW^*-algebra. Of course the definition and the characterization of a GW^*-algebra will play an important role.

In what follows, we suppose that $\mathcal{M}, \mathcal{N}$ are two closed $\mathcal{O}^*$-algebras on $\mathcal{D}, \mathcal{E}$ in $\mathcal{H}, \mathcal{K}$, respectively, such that $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$ and $\mathcal{N}'_w\mathcal{E} \subset \mathcal{E}$. From this follows that $\mathcal{M}'_w, \mathcal{N}'_w$ are von Neumann algebras. A natural question is what is the weak commutant of $\mathcal{M}\widetilde{\otimes}\mathcal{N}$?

Under our assumptions we always have that

$$\mathcal{M}'_w\overline{\otimes}\mathcal{N}'_w \subset (\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w.$$

To get the reverse inclusion, suppose that $(\overline{\mathcal{M}_b})'' = (\mathcal{M}'_w)'$ and $(\overline{\mathcal{N}_b})'' = (\mathcal{N}'_w)'$. In conclusion, the equality

$$\mathcal{M}'_w\overline{\otimes}\mathcal{N}'_w = (\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w,$$

holds for any EW^*-algebras $\mathcal{M}, \mathcal{N}$. But, we do not know whether the inclusion

$$(\mathcal{M} \otimes \mathcal{N})'_w(\mathcal{D}\widetilde{\otimes}\mathcal{E}) \subset \mathcal{D}\widetilde{\otimes}\mathcal{E}$$

holds true. If we suppose it holds, then the same is true for $\mathcal{M}\widetilde{\otimes}\mathcal{N}$, which implies that $((\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w)'$ is a von Neumann algebra. Thus, we may put:

- $\mathcal{M} \overset{GW^*}{\otimes} \mathcal{N} = \{a \in \mathcal{L}^\dagger(\mathcal{D}\widetilde{\otimes}\mathcal{E}) : \overline{a}\eta((\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w)'\}$.

This we call *GW^*-tensor product* of $\mathcal{M}, \mathcal{N}$, but because of our previous assumption, this does not always exist.

Taking again two closed $\mathcal{O}^*$-algebras $\mathcal{M}, \mathcal{N}$ such that $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$ and $\mathcal{N}'_w\mathcal{E} \subset \mathcal{E}$ we may define another "GW^*-tensor product" of $\mathcal{M}, \mathcal{N}$, which always exist and reads as follows:

- $(\mathcal{M}'_w)' \overset{\mathrm{GW}^*}{\otimes} (\mathcal{N}'_w)' = \{a \in \mathcal{L}^\dagger(\mathcal{D}\widetilde{\otimes}\mathcal{E}) : \overline{a}\eta(\mathcal{M}'_w)'\overline{\otimes}(\mathcal{N}'_w)'\}$.

We prove that $(\mathcal{M}'_w)' \overset{\mathrm{GW}^*}{\otimes} (\mathcal{N}'_w)'$ is a GW^*-algebra containing $\mathcal{M}\widetilde{\otimes}\mathcal{N}$ and we call it *GW^*-tensor product of $\mathcal{M}, \mathcal{N}$ defined by $(\mathcal{M}'_w)'$ and $(\mathcal{N}'_w)'$*. Note that if, $\mathcal{M}'_w\overline{\otimes}\mathcal{N}'_w = (\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w$, then the two GW^*-tensor products we have defined, coincide.

Finally we proceed to a more general definition of a GW^*-tensor product and we prove that all three of them coincide when $\mathcal{M}, \mathcal{N}$ are EW^*-algebras. In this case, we start with two von Neumann algebras $\mathcal{M}_0, \mathcal{N}_0$ over the Hilbert spaces $\mathcal{H}, \mathcal{K}$ and we suppose that there is a dense subspace $\mathcal{F}$ of the Hilbert space $\mathcal{H} \dot{\otimes} \mathcal{K}$, invariant under $(\mathcal{M}_0\overline{\otimes}\mathcal{N}_0)'$. Then,

$$\mathcal{A} := \{a \in \mathcal{L}^\dagger(\mathcal{F}) : \overline{a}\eta\mathcal{M}_0\overline{\otimes}\mathcal{N}_0\}$$

is an $\mathcal{O}^*$-algebra on $\mathcal{F}$ in $\mathcal{H} \dot{\otimes} \mathcal{K}$ and its closure $\widetilde{\mathcal{A}}$ is a GW^*–algebra on $\widetilde{\mathcal{F}} = \bigcap_{a\in\mathcal{A}} \mathcal{D}(\overline{a})$ in $\mathcal{H} \dot{\otimes} \mathcal{K}$ over $\mathcal{M}_0\overline{\otimes}\mathcal{N}_0$, and it is denoted by $\mathcal{M}_0 \overset{GW^*}{\underset{\mathcal{F}}{\otimes}} \mathcal{N}_0$. Hence,

- $\mathcal{M}_0 \overset{GW^*}{\underset{\mathcal{F}}{\otimes}} \mathcal{N}_0 := \widetilde{\mathcal{A}} := \{\widetilde{a} : a \in \mathcal{A}\}, \quad \widetilde{a} = \overline{a} \upharpoonright \cap_{a\in\mathcal{A}} \mathcal{D}(\overline{a}),$

and this is called *GW^*-tensor product defined by $\mathcal{M}_0$, $\mathcal{N}_0$ and $\mathcal{F}$*.

It is clear now that if $\mathcal{M}, \mathcal{N}$ are two closed $\mathcal{O}^*$-algebras on $\mathcal{D}, \mathcal{E}$ in $\mathcal{H}, \mathcal{K}$, respectively, such that $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$ and $\mathcal{N}'_w\mathcal{E} \subset \mathcal{E}$, then setting in the place of $\mathcal{M}_0, \mathcal{N}_0$ the von Neumann algebras $(\mathcal{M}'_w)'$, $(\mathcal{N}'_w)'$, respectively, and $\mathcal{D}\widetilde{\otimes}\mathcal{E}$ in the place of $\mathcal{F}$, we get

$$(\mathcal{M}'_w)' \overset{GW^*}{\otimes} (\mathcal{N}'_w)' = (\mathcal{M}'_w)' \overset{GW^*}{\underset{\mathcal{D}\widetilde{\otimes}\mathcal{E}}{\otimes}} (\mathcal{N}'_w)'$$

and if moreover $(\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w = \mathcal{M}'_w\overline{\otimes}\mathcal{N}'_w$, then

$$\mathcal{M} \overset{\mathrm{GW}^*}{\otimes} \mathcal{N} = (\mathcal{M}'_w)' \overset{\mathrm{GW}^*}{\otimes} (\mathcal{N}'_w)' = (\mathcal{M}'_w)' \overset{GW^*}{\underset{\mathcal{D}\widetilde{\otimes}\mathcal{E}}{\otimes}} (\mathcal{N}'_w)'.$$

Summing up, we have that in the following case the GW^*-tensor product is unique:

Proposition. *Let $\mathcal{M}$ and $\mathcal{N}$ be closed O^*–algebras on $\mathcal{D}$ and $\mathcal{E}$ respectively. If either $\overline{\mathcal{M}_b}' = \mathcal{M}'_w$ and $\overline{\mathcal{N}_b}' = \mathcal{N}'_w$, or if $\mathcal{M}$ and $\mathcal{N}$ are EW^*–algebras, then*

$$\mathcal{M} \overset{\mathrm{GW}^*}{\otimes} \mathcal{N} = (\mathcal{M}'_w)' \overset{\mathrm{GW}^*}{\otimes} (\mathcal{N}'_w)' = (\mathcal{M}'_w)' \overset{GW^*}{\underset{\mathcal{D}\widetilde{\otimes}\mathcal{E}}{\otimes}} (\mathcal{N}'_w)'.$$

III Applications

1. A GW^*-algebra $\mathcal{M}$ is called *properly W^*–infinite* if the von Neumann algebra $(\mathcal{M}'_w)'$ is properly infinite.

By the last property of $(\mathcal{M}'_w)'$ there exists a sequence $\{e_n\}$ of mutually orthogonal projections in it with $e_n \sim I_{\mathcal{H}}$, for all $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} e_n = I_{\mathcal{H}}$. Moreover, from $e_n \sim I_{\mathcal{H}}$, for all $n \in \mathbb{N}$, there exists a sequence $\{v_n\}$ of partial isometries in $(\mathcal{M}'_w)'$ such that

$$e_n = v_n^* v_n \text{ and } I_{\mathcal{H}} = v_n v_n^*, \ \forall \ n \in \mathbb{N}.$$

In this respect, we have the following:

Theorem. *Let $\mathcal{M}$ be a properly W^*–infinite GW^*–algebra on a Fréchet domain $\mathcal{D}$ in $\mathcal{H}$. Suppose that the graph topology $\tau_{\mathcal{M}}$ on $\mathcal{D}$ is defined by a sequence $\{\|\cdot\|_{t_n} : t_n \in \mathcal{M}\}$ of seminorms such that $v_n \overline{t_k} \subset \overline{t_k} v_n$, for all $k, n \in \mathbb{N}$. Then,*

$$\mathcal{M} \simeq \mathcal{M} \overset{\mathrm{GW}^*}{\otimes} \mathcal{B}(\mathcal{K}),$$

for every separable Hilbert space $\mathcal{K}$.

2. In a forthcoming paper joint with A. Inoue and K. Kürsten, we use GW^*-tensor products to study and construct crossed products of unbounded operator algebras.

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