# Endomorphisms of commutative semiprime Banach algebras

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# Abstract

This talk concerns joint work with Professor H. Kamowitz on compact, power compact, Riesz, and quasicompact endomorphisms of commutative Banach algebras.

We have shown that if $B$ is a unital commutative semisimple Banach algebra with connected character space, and $T$ is a unital endomorphism of $B$, then $T$ is quasicompact if and only if the operators $T^n$ converge in operator norm to a rank-one unital endomorphism of $B$.

We have extended these results in two ways.

- We have obtained results for endomorphisms of commutative Banach algebras which are semiprime and not necessarily semisimple.
- We also have results for commutative Banach algebras with character spaces which are not necessarily connected.

We shall discuss these results and their background, along with some related examples and questions.

# 1 Preliminaries

We shall only consider complex Banach algebras in this talk.

Let $A$ be a commutative, unital Banach algebra.

We denote by $\Phi_A$ the character space of $A$, and, for $a \in A$, we denote by $\hat{a}$ the Gelfand transform of $a$.

An (algebra) endomorphism $T$ of $A$ is called **unital** if $T1 = 1$.

If $A$ is semisimple, it follows from very early theory that endomorphisms of $A$ are automatically continuous.

In this case, $A$ may be regarded (via the Gelfand transform) as a **Banach function algebra** on $\Phi_A$.

Every unital endomorphism $T$ of $A$ then has the form $f \mapsto f \circ \phi$, where $\phi$ is the continuous selfmap $T^*|_{\Phi_A}$ of $\Phi_A$.

Even if $A$ is not semisimple, each unital endomorphism $T$ of $A$ still induces a selfmap $\phi$ of $\Phi_A$, which we call the **selfmap of $\Phi_A$ associated with $T$**.

Note that then, for all $a \in A$, we have

$$\widehat{Ta} = \hat{a} \circ \phi \, .$$

but we do not expect to be able to recover the endomorphism $T$ from $\phi$.

It is interesting to see what can be deduced about $T$ and the selfmap $\phi$ if you impose various operator theoretic conditions on $T$.

Much of the early work concerned compact endomorphisms of Banach function algebras, but more recently we (and others) have studied wider classes of endomorphisms.

We have also extended some of our results to cover **continuous** endomorphisms of commutative, semiprime Banach algebras.

# 2 Three noteworthy papers on compact endomorphisms

We now discuss some of the early work on compact endomorphisms.

- Herbert Kamowitz 1978 (disc algebra), 1980 (Banach function algebras)
- Udo Klein 1996 (120 pages, uniform algebras only)

# 3 Wider classes of operators

We now introduce the more general classes of operators we wish to discuss.

**Notation** Let $E$ be a Banach space.

We denote the sets of bounded operators and compact operators on $E$ by, respectively, $\mathcal{B}(E)$ and $\mathcal{K}(E)$.

**Definition 3.1** For a bounded operator $T$ on an infinite-dimensional Banach space $E$, the **essential spectrum** of $T$ is the set of complex numbers $\lambda$ such that $\lambda I - T$ is not a Fredholm operator.

This is also equal to the spectrum of $T + \mathcal{K}(E)$ in the Calkin algebra $\mathcal{B}(E)/\mathcal{K}(E)$.

Accordingly, the **essential spectral radius** of $T$, denoted by $r_e(T)$, is given by the formula

$$r_e(T) = \lim_{n\to\infty} \operatorname{dist}(T^n, \mathcal{K}(E))^{1/n}$$

$$= \inf\{\operatorname{dist}(T^n, \mathcal{K}(E))^{1/n} : n \in \mathbb{N}\}.$$

We say that $T$ is a **Riesz operator** if $\lambda I - T$ is Fredholm for all non-zero complex numbers $\lambda$.

Thus $T$ is Riesz if and only if $r_e(T) = 0$.

We say that $T$ is **quasicompact** if $r_e(T) < 1$.

Thus $T$ is quasicompact if and only if there exists $n \in \mathbb{N}$ with $\operatorname{dist}(T^n, \mathcal{K}(E)) < 1$.

An operator $T$ on a Banach space is said to be **power compact** if there exists a positive integer $N$ such that $T^N$ is compact.

Of course we have the following implications:

compact $\Longrightarrow$ power compact $\Longrightarrow$ Riesz $\Longrightarrow$ quasicompact.

In general, none of these implications can be reversed, even for endomorphisms of uniform algebras.

**Examples.**

# 4 General quasicompact endomorphisms

**Theorem 4.1 (F. – Kamowitz, 2005)** Let $B$ be a unital commutative semisimple Banach algebra with connected character space, let $T$ be a unital endomorphism of $B$, and let $\phi$ be the associated selfmap of $\Phi_B$.

Then $T$ is quasicompact if and only if the operators $T^n$ converge in operator norm to a rank-one unital endomorphism of $B$; in this case $\phi$ has a unique fixed point $x_0 \in \Phi_A$, and the rank-one endomorphism above must be the endomorphism $b \mapsto \hat{b}(x_0)1$.

**Theorem 4.2 (F. – Kamowitz, 2009)** Theorem 4.1 is valid for bounded unital endomorphisms of commutative semiprime Banach algebras whose character space is connected.

**Theorem 4.3 (F. – Kamowitz, 2009)** Let $B$ be a unital commutative semiprime Banach algebra, and let $T$ be a bounded unital endomorphism of $B$.

Then $T$ is quasicompact if and only if there is a natural number $n$ such that the operators $(T^{kn})_{k=1}^{\infty}$ converge in operator norm to a finite-rank unital endomorphism of $B$.

This result extends the results above for commutative semisimple Banach algebras, and results for uniform algebras of Klein [1996] and Gamelin, Galindo and Lindström [2004, 2007].

# 5 Questions

1. Choose your favourite Banach function algebra, and determine which selfmaps $\phi$ induce endomorphisms, compact endomorphisms , . . .

   (Quite a few people are now working in this area, including my current research student T. Chaobankoh.)

2. Solve similar problems for commutative semiprime Banach algebras.

3. In as much as you can solve these problems, can you determine the spectra of the associated endomorphisms (e.g. in terms of $\phi$)?

4. How much can you say about the nature of an endomorphism $T$ just by looking at the image $\phi(\Phi_A)$ of the associated selfmap $\phi$, or perhaps the set of images of the iterates of $\phi$?

5. Which other classes of operators are worth looking at?

6. What about non-commutative Banach algebras?