Simplicial cohomology of $l^1(\mathbb{N})$ and other semi-group algebras

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Introduction

Let $\mathcal{A}$ be a Banach algebra, $\mathcal{A}^*$ the dual of $\mathcal{A}$. Let $n \geqslant 1$ and $C_n(\mathcal{A}, \mathcal{A})$ the Banach space generated by $\{a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}, a_k \in \mathcal{A} \text{ and } 1 \leqslant k \leqslant n+1\}$. Let

$$\begin{aligned} d^n : C_{n+1}(\mathcal{A}, \mathcal{A}) &\longrightarrow C_n(\mathcal{A}, \mathcal{A}) \\ X = a_1 \otimes a_2 \otimes \cdots \otimes a_{n+2} &\longrightarrow \sum_{k=0}^{k=n+1} d_k^n(X), \end{aligned}$$

where

$$\begin{aligned} d_0^n(X) &= a_2 \otimes a_3 \otimes \cdots \otimes a_{n+1} \otimes a_{n+2}a_1, \\ d_k^n(X) &= (-1)^k a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_{n+2}, \text{ if } 1 \leqslant \mathrm{k} \leqslant \mathrm{n}+1, \end{aligned}$$

Introduction

$$C_{n-1}(\mathcal{A},\mathcal{A}) \leftarrow^{d^{n-1}} C_n(\mathcal{A},\mathcal{A}) \leftarrow^{d^n} C_{n+1}(\mathcal{A},\mathcal{A}).$$

We have that $d^{n-1} \circ d^n(X) = 0$, for all $X \in C_n(\mathcal{A},\mathcal{A})$.

Let $\delta^n = (d^n)^*$ and $(C_n(\mathcal{A},\mathcal{A}))^* = C^n(\mathcal{A},\mathcal{A}^*)$, then $\delta^n \circ \delta^{n-1} = 0$.

Thus we get the following chain

$$C^{n-1}(\mathcal{A},\mathcal{A}^*) \rightarrow^{\delta^{n-1}} C^n(\mathcal{A},\mathcal{A}^*) \rightarrow^{\delta^n} C^{n+1}(\mathcal{A},\mathcal{A}^*).$$

Introduction

Let $\Phi \in ker\delta^n$. Does there exist $\psi \in C^{n-1}(\mathcal{A}, \mathcal{A}^*) : \delta^{n-1}\psi = \Phi$?

Can we find a bounded linear operator

$$\sigma : C^n(\mathcal{A}, \mathcal{A}^*) \to C^{n-1}(\mathcal{A}, \mathcal{A}^*) :$$

$$\Phi = \delta^{n-1}\sigma\Phi$$

Introduction

As we have $\delta^n = (d^n)^*$ and $C^n(\mathcal{A}, \mathcal{A}^*) = (C_n(\mathcal{A}, \mathcal{A}))^*$,it suffices to show that there exist a bounded linear operator $s^p : C_p(\mathcal{A}, \mathcal{A}) \longrightarrow C_{p+1}(\mathcal{A}, \mathcal{A})$, for $p \in \{n, n-1\}$,

$$I(X) = (s^{n-1}d^{n-1} + d^n s^n)(X)$$

for all $X \in C_n(\mathcal{A}, \mathcal{A})$, where I is identity of the space.
To reach the result, we take $\sigma^p = (s^p)^*$, for $p \in \{n-1, n\}$, then we obtain

$$\phi = \delta^{n-1}(\sigma^{n-1}\phi).$$

Cohomology of l^1(semilattice)

In the paper **[C G W 2010]** Y.Choi et al showed that simplicial cohomology groups $\mathcal{H}^n(\mathcal{A}, \mathcal{A}^*)$ vanish for all $n \geqslant 2$, where $\mathcal{A}$ is a Band semigoup algebra, $\mathcal{A} = l^1(S)$, (S is a Band semigroup if S is a semigroup and $ss = s$, for all $s \in S$).

To do so, they determined the cyclic cohomology of $\mathcal{A}$, $\mathcal{HC}^n(\mathcal{A}, \mathcal{A}^*)$, and used the Connes-Tzygan long exact sequence to deduce the result for $\mathcal{H}^n(\mathcal{A}, \mathcal{A}^*)$.

Cohomology of l^1(semilattice)

As part of my PhD work (supervisor : F. Gourdeau), I have succeeded in obtaining $\mathcal{H}^n(\mathcal{A}, \mathcal{A}^*) = 0$, for all $n \geqslant 2$, without using cyclic cohomology.

To save time, I'll present the proof for particular case S that is a semilattice, (S is a commutative Band semigroup).

Cohomology of l^1(semilattice)

Definition

$s, t \in S$:

$$s \leqslant t \Leftrightarrow st = s$$

Choi et al showed the result through two steps.

Cohomology of l^1(semilattice), First step

We denote δ_s by s, where $s \in S$.

Definition

Let $X = s_1 \otimes s_2 \otimes \cdots \otimes s_{n+1} \in C_n(\mathcal{A}, \mathcal{A})$. We say that X has a minimal element, say s_{k_0} for $1 \leqslant k_0 \leqslant n+1$, if $s_{k_0} \leqslant s_k$ for all $1 \leqslant k \leqslant n+1$.

In this step they showed that there exist s such that $\big(sd + ds\big)(X) =$

$$X + \sum_{m=1}^{p_X} Y_m$$

if $X \neq (x \otimes x \otimes \cdots \otimes x)$ and

$$0$$

else, where X has at least one minimal element,

Cohomology of l^1(semilattice), First step

And Y_m has strictly more minimal element than X.

As X has at most $n+1$ minimal element, by repeating the algorithm $(n+1)$-times, we get a finite combination of form $(x \otimes x \otimes \cdots \otimes x)$. Later, they showed that we can cobound in this space.

At this step, we can omit the word cyclic to get the same result in the simplicial homology.

Cohomology of l^1(semilattice), Second step

The purpose of this part is to find a suitable application Q such that $Q(X)$ is a finite linear combination of elements which admit at least one minimal element.

In this section we will also need some definitions.

Cohomology of l^1(semilattice), Second step

Definition

Let $X = s_1 \otimes \cdots \otimes s_{n+1}$ be without minimal element. We say that a subtensor $s_k \otimes \cdots \otimes s_l$ has a minimal left element if $s_k \leqslant s_i$ for all i in the cyclic interval $[k, l]$. A subtensor is a left-block if it has a minimal left element and is not strictly included in another subtensor which has a minimal left element.

For elementary tensor X without minimal element, it is easy to see that it has a unique decomposition into left-blocks, and therefore we can define

$$I_X = \{i \in \{1, 2, \ldots, n+1\} : x_i \text{ is the first component of a left-block}\}$$

If X has a minimal element, $I_X = \varnothing$.

Cohomology of l^1(semilattice), Second step

Definition

If T is a finite semilattice, and $\alpha \in T$, the *height of* α *in* T, denoted by $h_T(\alpha)$, is the length of the longest descending chain in T which starts at α.

If $X = x_1 \otimes \cdots \otimes x_{n+1}$ is an elementary tensor, let $L(X)$ be the subsemilattice that is generated by $\{x_1, \ldots, x_{n+1}\}$ and we define the *height* of X to be

$$h(X) = \sum_{i=1}^{n+1} h_{L(X)}(x_i).$$

It's important to see that if X has no minimal element then

$$n + 1 \leqslant h(X) \leqslant n(n+1).$$

Cohomology of l^1(semilattice), Second step

Now, we can give a bounded linear map s. Let X an elementary tensor, and

$$s(X) = \sum_{i \in I_X} (-1)^i x_1 \otimes \cdots \otimes x_i \otimes x_i \otimes \cdots \otimes x_{n+1}.$$

Then

$$(s^{n-1}d^{n-1} + d^n s^n)(X) = kX + \sum_{i=1}^{p} Y_i + \sum_{j=1}^{q} Z_j + \begin{cases} t(X) & \text{if } x_1 < x_{n+1} \\ 0 & \text{else} \end{cases}$$

where

- Y_i has strictly less left blocks than X,
- $h(Z_j) < h(X)$,
- $1 \leqslant k \leqslant n+1$,
- $t(X) = (-1)^n(x_{n+1} \otimes x_1 \otimes \cdots \otimes x_n)$.

Cohomology of l^1(semilattice), Second step

Let $P_r = I - \frac{1}{r}(sd + ds)$ for $r > 0$. Then

$$P_k^{n-1}(X) = \sum_{i=1}^{p} V_i + \sum_{j=1}^{q} W_j + \begin{cases} t^{n-1}(X) & \text{if } x_1 < x_{n+1} < x_n < \cdots < x_3 \\ 0 & \text{else} \end{cases}$$

where V_i has strictly less left blocks than X and $h(W_j) < h(X)$. If in this case X has no minimal element, then t does not occur in the next iteration.

Therefore, for $P = P_1 \circ \cdots \circ P_{n+1}$, we have $P^n(X)$ is a finite sum of elements which have strictly less left blocks than X or elements of height strictly smaller than the height of X

Cohomology of l^1(semilattice), Second step

Remember that if X has no minimal element then

$$n + 1 \leqslant h(X) \leqslant n(n + 1)$$

and X has at most $n + 1$ left-block. Then for X an elementary tensor, we get

$P^{n^3}(X)$ is a finite sum of elements that have at least one minimal elelment. Which completes the work.

Cohomology of $l^1(\mathbb{Z}_+)$

I managed to not use cyclic cohomology algebra in another algebra : $l^1(\mathbb{Z}_+)$.

Cohomology of $l^1(\mathbb{Z}_+)$

Let $\mathcal{A} = l^1(\mathbb{Z}_+)$ be the unital semigroup algebra. By **[G. J. W, 2005]** we know that $\mathcal{H}^n(\mathcal{A}, \mathcal{A}^*)$ vanish when $n \geqslant 2$.

To do so, **Gourdeau et al** determined the cyclic cohomology of $\mathcal{A}$, $\mathcal{HC}^n(\mathcal{A}, \mathcal{A}^*)$ and used the Connes-Tzygan long exact sequence to deduce the result for $\mathcal{H}^n(\mathcal{A}, \mathcal{A}^*)$. In the fifth section of this paper the authors gave an explicit formula for the contracting homotopy in the cyclic case.

Cohomology of $l^1(\mathbb{Z}_+)$

We give an explicit formula for the contracting homotopy in simplicial homology.

$$\rho(m) = \sum_{j=0}^{m-1} m - j \otimes j$$

Let $s^1, s^2 : C_n(\mathcal{A}, \mathcal{A}) \to C_{n+1}(\mathcal{A}, \mathcal{A})$ where

$s^1(m_1 \otimes \cdots \otimes m_{n+1}) =$

$$\sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{N}(m_1 \otimes \cdots \otimes \rho(m_k) \otimes \cdots \otimes m_{n+1})$$

if $N \neq 0$, and

$$(0 \otimes 0 \otimes \cdots \otimes 0 \otimes 0)$$

if $N = 0$.

Cohomology of $l^1(\mathbb{Z}_+)$

$$s^2(m_1 \otimes \cdots \otimes m_{n+1}) =$$

$$\frac{(-1)^{n+1}}{N} \sum_{j=0}^{m_1-1} (j \otimes m_2 \otimes \cdots \otimes m_n \otimes m_{n+1} + m_1 - j \otimes 0)$$

if $N \neq 0$, and

$$(0 \otimes \cdots \otimes 0 \otimes 0)$$

if $N = 0$, and

$$N = m_1 + \cdots + m_{n+1}$$

Cohomology of $l^1(\mathbb{Z}_+)$

$$
\begin{aligned}
(s^1 d + ds^1)(X) &= X \\
&+ \Big[\frac{(-1)^n}{N} m_2 \otimes \cdots \otimes m_n \otimes \big(m_{n+1} + \rho(m_1)\big)\Big] \\
&+ \Big[\frac{-1}{N} d_0(\rho(m_1) \otimes m_2 \otimes \cdots \otimes m_{n+1})\Big]
\end{aligned}
$$

Cohomology of $l^1(\mathbb{Z}_+)$

$$\begin{aligned}
s^2 d + d s^2 &= s^2 d_0 + d_0 s^2 + \\
&\quad s^2 d_1 + d_1 s^2 + \\
&\quad s^2 d_2 + d_2 s^2 \\
&\quad \vdots \\
&\quad s^2 d_n + d_n s^2 + \\
&\quad d_{n+1} s^2 \\
&= d_0 s^2 + d_{n+1} s^2
\end{aligned}$$

Because $s^2 d_0 + s^2 d_1 + d_1 s^2 = 0$ and $s^2 d_k + d_k s^2 = 0$, for $2 \leqslant k \leqslant n$.

Cohomology of $l^1(\mathbb{Z}_+)$

As

$$d_0 s^2(X) = \frac{(-1)^{n+1}}{N} \sum_{j=0}^{m_1-1} m_2 \otimes \cdots \otimes m_{n+1} + m_1 - j \otimes j$$
$$= \frac{-(-1)^n}{N} m_2 \otimes \cdots \otimes m_n \otimes m_{n+1} + \rho(m_1),$$

$$d_{n+1} s^2(X) = \frac{1}{N} \sum_{j=0}^{m_1-1} j \otimes m_2 \otimes \cdots \otimes m_{n+1} + m_1 - j$$
$$= \frac{1}{N} d_0(\rho(m_1) \otimes m_2 \otimes \cdots \otimes m_{n+1})$$

We conclude that

$$\Big((s^1 + s^2)d + d(s^1 + s^2)\Big)(X) = X$$

Generalization to other ordered discrete semigroup

Let $\mathcal{G}$ be a subgroup of $\mathbb{R}$ with the discrete topology, let $\mathcal{S} = \mathcal{G} \bigcap \mathbb{R}_+$. Let $\mathcal{B} = l^1(\mathcal{S})$.

By **[G J W 2005]** we know that, for all $n \geqslant 2$:

$$\mathcal{H}^n(\mathcal{B}, \mathcal{B}^*) = 0$$

The proof uses Connes-Tzygan. The contracting homotopy, in cyclic cohomology is given by

$$\sigma_1 T(m_1, \ldots, m_n)(m_{n+1}) =$$

$$\sum_{k=1}^{n+1} (-1)^{k+1} \mu\big(T[m_1, \ldots, m_k - x, x, \ldots, m_{n+1}]_0^{a_k}\big)$$

Generalization to other ordered discrete semigroup

where μ is an invariant mean on the amenable group $\mathcal{S}_{\mathcal{N}} = \{s \in \mathcal{S} : 0 \leqslant s < N\}$ with the addition is mod N, and

$$N = m_1 + \cdots + m_{n+1}$$

$T[m_1, \ldots, m_k - x, x, \ldots, m_{n+1}]_0^{a_k} \in l^\infty(\mathcal{S}_{\mathcal{N}})$ defined by

$$T[m_1, \ldots, m_k - x, x, \ldots, m_{n+1}]_0^{a_k}(s) =$$

$$\begin{cases} T(m_1, \ldots, m_k - s, s, \ldots, m_{n+1}) & \text{if } 0 \leqslant s < a_k \\ 0 & \text{otherwise} \end{cases}$$

Generalization to other ordered discrete semigroup

The contracting homotopy, in the simplicial cohomology is given by

$$\sigma = \sigma_1 + \sigma_2$$

where

$$\sigma_2 T(m_1, \ldots, m_n)(m_{n+1}) =$$
$$(-1)^{n+1} \mu\big(T[x, m_2, \ldots, m_{n+1} + m_1 - x, 0]_0^{m_1} \big)$$