On West Compactifications of Locally Compact Abelian Groups

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Outline

G: loc. cpct gp.

- (ψ, X) is a *semigroup compactification* (sgr cpctf) if
  - X is a cpct, Hdf, right topological semigroup;
  - $\psi : G \to X$ is a cts. homomorphism;
  - $\psi(G)$ is dense in X;
  - $\psi(G)$ is in the *topological center* $\Lambda(X) = \{t \in X : \text{The func. } X \to X : s \to ts \text{ is cts.}\}$.
- Classify cpctfs wrt:
  - algebraic/ topological properties of X *e.g.* X: topological group, semitopological semigroup.
  - properties of $\mathcal{C}(X)|_G \subset \mathcal{C}_b(G)$ *e.g.* $\mathcal{C}(X)|_G \subset AP(G)$ or $\mathcal{C}_0(G)$.

- *e.g.* NO cpctf (ψ, X) of $\mathbb{R}$ satisfies $\mathcal{C}(X)|_{\mathbb{R}} = \mathcal{C}_b(\mathbb{R})$.

- **Thm1.** For any subalgebra, $\mathcal{A}$ of $\mathcal{C}_b(G)$ which is
    - norm closed
    - conjugate closed
    - translation invariant
    - contains the constants
    - invariant under introversion operators determined by multiplicative linear functionals on $\mathcal{A}$

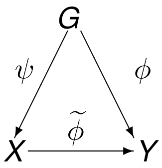
  $(\epsilon, \sigma(\mathcal{A}))$ gives a sgr cpctf of G with $\mathcal{C}(X)|_G = \mathcal{A}$.

- If $\mathcal{C}(X)|_G \subset \mathcal{A}$, then X is called an $\mathcal{A}$-compactification,
- If $\mathcal{C}(X)|_G = \mathcal{A}$, then X is called the Universal $\mathcal{A}$-compactification.

Order on Cpctfs of G

Let (ψ, X) and (ϕ, Y) be cpctfs of G.

- If $\exists$ cts. hom. $\widetilde{\phi} : X \to Y$



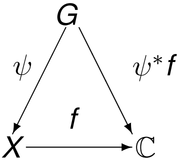
Y is a quotient of X, $Y \preceq X$.

- Y is called a *factor* of X.
- In this case $C(Y)|_G \subset C(X)|_G$.

Dual Map

Given $\psi : G \to X$ a sgr cpctf

- Define the dual map $\psi^* : \mathcal{C}(X) \to \mathcal{C}_b(G)$, of ψ by



- If $G \subset X$, ψ^* is the restriction map.
- $\phi^*\mathcal{C}(Y) \subset \psi^*\mathcal{C}(X) \Rightarrow Y$ is a factor of X .
- Hence any $\mathcal{A}$-compactification is a quotient of the Universal $\mathcal{A}$-compactification.

$WAP(G)$

- Let $f \in \mathcal{C}_b(G)$. The *orbit* of f is defined by

$$O(f) = \{f_g = f(g.) : g \in G\},$$

- We call f a **weakly almost periodic function** (**w.a.p**) if the $O(f)$ is relatively weakly compact.
- $WAP(G)$ denotes the set of w.a.p functions on G.
- **Grothendieck Criterion**: A cts. func. is w.a.p iff for any two sequences $\{t_n\}$ and $\{s_m\} \subset G$

$$\lim_m \lim_n f(t_n s_m) = \lim_n \lim_m f(t_n s_m)$$

whenever both limits exist.

Properties of $WAP(G)$

- $WAP(G)$ satisfy the criteria of Thm1, so the w.a.p cpctf of G exists, denoted by (ψ, G^w).
- Grothendieck Criterion $\Rightarrow$ G^w is a cpct semitopological semigroup.
- G^w is also universal among (ψ, X) s.t. X is semitopological sgr.
- $\mathbb{C} \cup \mathcal{C}_0(G) \subset WAP(G) \Rightarrow G^\infty \preceq G^w$.
- $\psi : G \to G^w$ is a homeomorphism.

B(G)

- Let $\mathcal{U}(\mathcal{H})$ be the *unitary operators* on Hilbert sp $\mathcal{H}$
  A **unitary representation** of G is a homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$, cts wrt the **SOT**.
- $\Sigma = \{$Equiv. classes of cts unitary repn's of $G\}$.
- $B(G) = \{g \to \langle \pi(g)\xi, \eta\rangle : \pi \in \Sigma, \xi, \eta \in \mathcal{H}_\pi\}$
  - $B(G)$ is a subalgebra of $\mathcal{C}_b(G)$
  - $B(G) = C^*(G)^*$ via

$$< f, u >= \int_G f(x)u(x)dx, \; u \in B(G) \; f \in L^1(G)$$

  $B(G)$ is a comm. unital Ban algebra.
- [Bochner] G abelian $\Rightarrow B(G) = \mathcal{FS}(M(G))$.

$\mathcal{E}(G)$

- [Eberlein] $B(G) \subset WAP(G)$.
- $B(G)$ satisfies the prop in **Thm1**, but $B(G)$ is NOT uniformly closed.
- $\mathcal{E}(G) = \overline{B(G)}^{\|\cdot\|_\infty}$, called the **Eberlein algebra**
  - $\exists$ Corresp. Universal cpctfn, denoted by (ϕ, G^e).
  - $\mathcal{E}(G) \subset WAP(G) \;\Rightarrow G^e$ is a stpl sgr and $G^e \preceq G^w$.
  - $C_0(G) \subset \mathcal{E}(G) \;\Rightarrow\; \phi$ is a homeomorphism.

The relation "$\mathcal{E}(G) \subset WAP(G)$"

- G cpct $\Rightarrow \mathcal{E}(G) = WAP(G) = \mathcal{C}(G)$.
- G non-compact, loc. cpct. gr.
  - [Chou] If $G = SL_2(\mathbb{R})$, then

$$WAP(SL_2(\mathbb{R})) = \mathbb{C} + \mathcal{C}_0(SL_2(\mathbb{R})) = \mathcal{E}(SL_2(\mathbb{R})).$$

  - [Mayer, Veech] For a larger class of semisimple Lie groups, we have $\mathcal{E}(G) = WAP(G)$
  - [Rudin] If $G = \mathbb{Z}$, then $WAP(G) \neq \mathcal{E}(G)$,
    [Ramirez] If G non-cpct, Abelian, then $WAP(G) \neq \mathcal{E}(G)$,
  - [Chou] If G non-cpct nilpotent/[IN]-gr, then the quotient $WAP(G)/\mathcal{E}(G)$ contains a linear isometric copy of l^∞.

We will construct a semigroup compactification for any l.c.a. G, that is a quotient of both G^e and G^w

- In 1958 Ellis proved that every *compact right − topological semigroup* contains an idempotent.
- In 1968 West constructed a *compact singly generated semitopological semigroup*, S, which contains 2 idempotents.
  - $\mathbb{Z}$ is dense in $S \Rightarrow S \preceq \mathbb{Z}^w$ and $S \preceq \mathbb{Z}^e$.

Goal:

- To generalize West's construction for any l.c.a group G;
- To characterize each compact semigroup, called a West semigroup corresponding to G, denoted by $\overline{G}^*$.

G loc. cpct. Abelian

- $\widehat{G} = \{\gamma : G \to \mathbb{T} : \gamma \text{ } \textit{cts gr homomorphism}\}$,
- $\widehat{G}$ with pointwise multiplication is the *Dual Group* of G.
- The dual of $\widehat{G}$ is G.
- In fact, the duality is given by Gelfand Transform.
- G is compact if and only if $\widehat{G}$ is discrete,
- We will study **non-discrete** lca groups $\widehat{G}$.

- $\widehat{G}$ is called an *I*-**group**, if every nbhd U of 1 in $\widehat{G}$ contains an element of infinite order.
- Example of an I-group: $\widehat{G} = \mathbb{T}$, where $G = \mathbb{Z}$
- Example of non-I-group: $\widehat{G} = \mathbb{D}_q$, $q \in \mathbb{N}^{\geq 2}$
  - If $G = \mathbb{Z}_q$, qth-roots of unity, $\widehat{G} = \mathbb{Z}_q$, where
  - $\widehat{G} = \mathbb{D}_q$ is the complete direct product of countable $\mathbb{Z}_q$, and $G = (\mathbb{Z}_q)^\infty$, with discrete topology.
- **Thm2.** If $\widehat{G}$ non-disc, Non-I-group, then it contains $\mathbb{D}_q$ as a closed subgroup.

- $E \subset \widehat{G}$ is called a *Cantor set* if E is metrizable, perfect and totally disconnected.
  - Equivalently, if E is homeomorphic to the classical Cantor subset, C, of $[0, 1]$.
- $E \subset \widehat{G}$ is a *Kronecker set* if $\forall f : E \to \mathbb{T}$ cts and $\forall \epsilon > 0$, $\exists \gamma \in G$ s.t
$$\|f - \gamma|_E\| < \epsilon$$
- For $q \geq 2$, $E \subset \widehat{G}$ is a K_q *set* if $\forall f : E \to \mathbb{Z}_q$ cts and $\forall \epsilon > 0$, $\exists \gamma \in G$ s.t
$$\|f - \gamma|_E\| < \epsilon$$

Thm3.

- If $\widehat{G}$ is an I-group, $\exists E \subset \widehat{G}$, a Cantor set which is also a Kronecker set.
- If $\widehat{G} = \mathbb{D}_q$, $\exists E \subset \widehat{G}$, a Cantor set which is also a K_q set.

- Consider $E \subset \widehat{G}$, which is *Cantor*, *Kronecker* (or K_q),
  - E Cantor $\Rightarrow \exists \mu_0 \in M_c^+(E), \mu_0 \neq 0$.
  - E Kronecker (or K_q) $\Rightarrow \mathcal{C}(E, \mathbb{T}) \subset \overline{G}^{\|\cdot\|}$ (or $\mathcal{C}(E, \mathbb{Z}_q) \subset \overline{G}^{\|\cdot\|}$).
- Let $\overline{G}^*$ be the weak*-closure of G in $L^\infty(E, \mu_0)$, called a **West Semigroup** of G

Thm.

- (i) When $\widehat{G}$ is an I-group, $\overline{G}^*$, is isomorphic to $(L^\infty)_1$;
- (ii) When $\widehat{G}$ is a non-discrete, non-I-group, $\overline{G}^*$, is isomorphic to $(L^\infty)_{S_q}$

- Let $(L^\infty)_1$ be the closed unit ball of $L^\infty([0, 1], \lambda)$;
- Let S_q be the closed convex hull of $\mathbb{Z}_q$, and $(L^\infty)_{S_q}$ be the S_q-valued functions in $L^\infty([0, 1], \lambda)$.

$\overline{G}^* \cong (L^\infty)_1$

- $L^\infty([0,1],\lambda) \cong L^\infty(E,\mu_0)$;
- $[0,1]$ is compact so S_1, the set of simple funcs w^*- dense in $(L^\infty)_1$;
- We need: $e^{i\theta}\chi_{[s,t)} \in \overline{G}^*$ for all $e^{i\theta} \in \mathbb{T}$, $t,s \in [0,1]$
- [West] $\chi_{[s,t)} \in w^*\text{-cl}\{f_{t,s}^n : n \in \mathbb{Z}\}$ where

$$f_{t,s}(x) = \begin{cases} 1, & \textit{if } 0 \leq x \leq t \\ e^{i(t-x)(s-x)}, & \textit{if } t \leq x \leq s \\ 1, & \textit{if } s \leq x \leq 1 \end{cases}$$

- $f_{t,s} \in \overline{G}^* \Rightarrow \chi_{[s,t)} \in \overline{G}^*$
- $e^{i\theta} \in \overline{G}^* \Rightarrow e^{i\theta}\chi_{[s,t)} \in \overline{G}^*$.

$\overline{G}^* \cong (L^\infty)_{S_q}$

- $L^\infty([0,1],\lambda) \cong L^\infty(E,\mu_0)$;
- We need: $e^{i\theta}\chi_{[s,t)} \in \overline{G}^*$ for all $e^{i\theta} \in \mathbb{Z}_q$, $t,s \in [0,1]$
- For $t,s \in E$ $t < s$ define for $x \in E$

$$f^1_{t,s}(x) = \begin{cases} 1, & \text{if } 0 \leq x < t \\ (\iota), & \text{if } t \leq x \leq s \\ 1, & \text{if } s < x \leq 1 \end{cases}$$

where

$$(\iota) = f^1_{t,s}(x) = \begin{cases} e^{2\pi i/q}, & \text{if } x \in S^1_1 \\ \vdots & \vdots \\ e^{2\pi i}, & \text{if } x \in S^1_q \end{cases}$$

Consequences

- $\overline{G}^*$ is a quotient of G^e and G^w.
- In both cases $I(\overline{G}^*) = (L^\infty)_{\{0,1\}} \Rightarrow |I(\overline{G}^*)| = \mathfrak{c}$.
  - [Brown & Moran ′72-′75] $|I(G^e)| \geq \rfloor \Rightarrow |I(G^w)| \geq \mathfrak{c}$.

    [Ruppert '91] $|I(\mathbb{Z}^w)| = 2^{\mathfrak{c}}$.
    [Pym '96] $|I((\mathbb{Z}_q^\infty)^w)| = 2^{\mathfrak{c}}$.

  Open Question: $|I(G^e)| =?$
- $\overline{I(\overline{G}^*)} = (L^\infty)_{[0,1]} \Rightarrow I(\overline{G}^*)$ is not closed.
  - [Lemanczyk, Bouziad & Mentzen ′00] $I(\mathbb{Z}^e)$ and $I(\mathbb{Z}^w)$ is not closed.
  - [Pym & Mentzen ′96] and $I((\mathbb{Z}_q^\infty)^w)$ is not closed.