Multi-norms, duality, and the injectivity of $L^p(G)$

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References

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Amenable Banach algebras

Let A be a Banach algebra, let E be a Banach A-bimodule, and let E' be the dual Banach A-bimodule.

Definition - Johnson A Banach algebra A is **amenable** if every continuous derivation

$$D : A \to E'$$

is inner for every Banach A-bimodule E.

Let $L^1(G)$ be the group algebra of a locally compact group G. Then of course:

Theorem - B. E. Johnson, 1972 The Banach algebra $L^1(G)$ is amenable if and only if the locally compact group G is amenable. $\square$

Injectivity of modules

Let A be an algebra, and let E and F be left A-modules. A linear map $T : E \to F$ is an A-**module morphism** if $T(a \cdot x) = a \cdot Tx$ for all $a \in A$ and $x \in E$.

Let A be a Banach algebra, and let E and F be Banach left A-modules. The space of bounded linear maps which are A-module morphisms is denoted by ${}_A\mathcal{B}(E, F)$. Such a morphism is **admissible** if there is a bounded linear map $S : F \to E$ with $S \circ T = I_E$, and a **coretraction** if there is such an S in ${}_A\mathcal{B}(F, E)$.

Let A be a Banach algebra, and let J be a Banach left A-module. Then J is **injective**, if, for each Banach left A-modules E and F, for each admissible homomorphism $T \in {}_A\mathcal{B}(E, F)$, and each $S \in {}_A\mathcal{B}(E, J)$, there exists $R \in {}_A\mathcal{B}(F, J)$ with $R \circ T = S$.

A condition for injectivity

Let A be a Banach algebra, and let E be a Banach space. For $T \in \mathcal{B}(A,E)$ and $a \in A$, set

$$(a \cdot T)(b) = T(ba) \quad (b \in A)\,.$$

Then $\mathcal{B}(A,E)$ is also a Banach left A-module. The **canonical embedding** $\Pi : E \to \mathcal{B}(A,E)$ is

$$\Pi(x)(a) = a \cdot x \quad (a \in A,\, x \in E)\,.$$

When E is a left A-module, $\Pi \in {}_A\mathcal{B}(E, \mathcal{B}(A,E))$.

Theorem Let A be a Banach algebra, and let J be a Banach left A-module. Then J is injective if and only if $\Pi \in {}_A\mathcal{B}(J, \mathcal{B}(A,J))$ is a coretraction. $\square$

So we need an A-module homomorphism $\rho : \mathcal{B}(A,J) \to J$ such that $\rho \circ \Pi = I_J$.

The injective (and projectivity) of Banach left $L^1(G)$-modules were studied in [**DP1**].

For example, $L^1(G)$ itself is injective iff G is discrete and amenable.

Helemski's theorem

Theorem - Helemski, 1984 Let A be an amenable Banach algebra. Then E' is injective for each Banach right A-module E. $\square$

Let G be a locally compact group. For $p \geq 1$,

$$L^p(G) = \left\{ f : \|f\|_p = \left(\int_G |f(s)|^p \, \mathrm{d}m(s) \right)^{1/p} < \infty \right\} .$$

Now $f \star g \in L^p(G)$ when $f \in L^1(G)$ and $g \in L^p(G)$, and $L^p(G)$ is a Banach left $L^1(G)$-module; for $p > 1$, it is the dual of the Banach right $L^1(G)$-module $L^q(G)$, where $q = p'$

Corollary Let G be an amenable group. Then $L^p(G)$ is an injective Banach left $L^1(G)$-module for each $p > 1$. $\square$

The following **conjecture** was raised in [**DP1**]: Suppose that $L^p(G)$ is an injective Banach left $L^1(G)$-module for some or each $p > 1$. Then G is amenable.

Basic definitions

Let $(E, \|\cdot\|)$ be a normed space.

Definition A **multi-norm** on $\{E^n : n \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_n)$ such that each $\|\cdot\|_n$ is a norm on E^n, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the following hold for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in E$:

(A1) $\left\|(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\right\|_n = \|(x_1, \ldots, x_n)\|_n$ for each permutation σ of $\{1, \ldots, n\}$;

(A2) $\|(\alpha_1 x_1, \ldots, \alpha_n x_n)\|_n$

$$\leq (\max_{i \in \mathbb{N}_n} |\alpha_i|)\, \|(x_1, \ldots, x_n)\|_n$$

for each $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$;

(A3) $\|(x_1, \ldots, x_n, 0)\|_{n+1} = \|(x_1, \ldots, x_n)\|_n$;

(A4) $\|(x_1, \ldots, x_n, x_n)\|_{n+1} = \|(x_1, \ldots, x_n)\|_n$.

Dual multi-norms

We get a **dual multi-norm** if we replace (A4) by (B4):

(B4) for each $x_1, \ldots, x_n \in E$, we have

$$\|(x_1, \ldots, x_n, x_n)\|_{n+1} = \|(x_1, \ldots, 2x_n)\|_n \, .$$

The dual of a multi-norm is a dual multi-norm, and the dual of a dual multi-norm is a multi-norm.

There is a more general notion of a 'special-norm' that also encompasses the 'sequential norms' of Lambert, Neufang, and Runde; there are connections with operator spaces.

Minimum and maximum multi-norms

Let $(\|\cdot\|_n : n \in \mathbb{N})$ be a multi-norm based on a Banach space E. Then

$$\max \|x_i\| \le \|(x_1, \ldots, x_n)\|_n \le \sum_{i=1}^{n} \|x_i\| \qquad (*)$$

for all $x_1, \ldots, x_n \in E$ and $n \in \mathbb{N}$.

Example 1 Set $\|(x_1, \ldots, x_n)\|_n^{\text{min}} = \max \|x_i\|$. This gives the **minimum** multi-norm.

Example 2 It follows from (*) that there is also a **maximum** multi-norm, which we call $(\|\cdot\|_n^{\text{max}} : n \in \mathbb{N})$.

Note that it is **not** true that $\sum_{i=1}^{n} \|x_i\|$ gives the maximum multi-norm — because it is not a multi-norm. In fact it gives a dual multi-norm.

A characterization

This is taken from [DDPR1]. It gives a 'coordinate-free' characterization.

Let $(E, \|\cdot\|)$ be a normed space. Then a c_0-**norm** on $c_0 \otimes E$ is a norm $\|\cdot\|$ such that:

1) $\|a \otimes x\| \leq \|a\| \, \|x\| \quad (a \in c_0, x \in E)$;

2) $T \otimes I_E$ is bounded on $(c_0 \otimes E, \|\cdot\|)$ with $\|T \otimes I_E\| = \|T\|$ whenever T is a compact operator on c_0;

3) $\|\delta_1 \otimes x\| = \|x\| \quad (x \in E)$;

Theorem Multi-norms on $\{E^n : n \in \mathbb{N}\}$ correspond to c_0-norms on $c_0 \otimes E$. The injective tensor product norm gives the minimum multi-norm, and the projective tensor product norm gives the maximum multi-norm $\square$

An associated sequence

Let $(\|\cdot\|_n)$ be a multi-norm on $\{E^n : n \in \mathbb{N}\}$.

Define

$$\varphi_n(E) = \sup\{\|(x_1, \dots, x_n)\|_n : \|x_i\| \leq 1\}\,.$$

Trivially, $1 \leq \varphi_n(E) \leq n$ for all $n \in \mathbb{N}$ and

$$\varphi_{m+n}(E) \leq \varphi_m(E) + \varphi_n(E)$$

for all $m, n \in \mathbb{N}$. The sequence $(\varphi_n(E))$ is the **rate-of-growth** of the multi-norm.

In particular $(\varphi_n^{\max}(E))$ is the sequence associated with the maximum multi-norm.

It can be shown quite easily that $\varphi_n^{\max}(E)$ is

$$\sup\left\{\sum_{j=1}^{n} \left\|\lambda_j\right\|\right\}\,,$$

where $\lambda_1, \dots, \lambda_n \in E'$ and

$$\sum_{j=1}^{n} \left|\langle x,\, \lambda_j\rangle\right| \leq 1 \quad (x \in E_{[1]})\,.$$

Special spaces

Take p with $1 \leq p \leq \infty$. Direct calculations of $\varphi_n^{\max}(\ell^p)$ using Banach–Mazur distance give:

Theorem (i) For each $p \in [1,2]$, we have

$$\varphi_n^{\max}(\ell_n^p) = \varphi_n^{\max}(\ell^p) = n^{1/p} \quad (n \in \mathbb{N}).$$

(ii) For each $p \in [2,\infty]$, there is a constant C_p such that

$$\sqrt{n} \leq \varphi_n^{\max}(\ell_n^p) = \varphi_n^{\max}(\ell^p) \leq C_p\sqrt{n} \quad (n \in \mathbb{N}).$$

$\square$

We have $C_2 = 1$ and $C_1 \leq \sqrt{2}$, but otherwise I do not know the best constant C_p in the above inequality.

From Dvoretsky's theorem, we have:

Theorem Let E be an infinite-dimensional normed space. Then $\varphi_n^{\max}(E) \geq \sqrt{n}$ for each $n \in \mathbb{N}$. $\square$

Summing norms - I

Let E be a normed space, and take $p \in [1, \infty)$. For $x_1, \dots, x_n \in E$, set

$$\mu_{p,n}(x_1, \dots, x_n) = \sup_{\lambda \in E'_{[1]}} \left\{ \left(\sum_{j=1}^{n} \left| \langle x_j, \lambda \rangle \right|^p \right)^{1/p} \right\} .$$

This is the **weak** p**-summing norm**. For example, we can see that

$$\mu_{1,n}(x_1, \dots, x_n) = \sup \left\{ \left\| \sum_{j=1}^{n} \zeta_j x_j \right\| : \zeta_1, \dots, \zeta_n \in \mathbb{T} \right\} .$$

For $\lambda_1, \dots, \lambda_n \in E'$, we have

$$\mu_{1,n}(\lambda_1, \dots, \lambda_n) = \sup \left\{ \sum_{j=1}^{n} \left| \langle x, \lambda_j \rangle \right| : x \in E_{[1]} \right\} .$$

Let E and F be Banach spaces, take p, q with $1 \le p \le q < \infty$, and take $T \in \mathcal{B}(E, F)$ and $n \in \mathbb{N}$. Then $\pi_{q,p}^{(n)}(T)$ is

$$\sup \left\{ \left(\sum_{j=1}^{n} \left\| T x_j \right\|^q \right)^{1/q} : \mu_{p,n}(x_1, \dots, x_n) \le 1 \right\} .$$

Summing norms - II

Again $1 \leq p \leq q < \infty$.

Definition Let $T \in \mathcal{B}(E,F)$. Suppose that

$$\pi_{q,p}(T) := \lim_{n\to\infty} \pi_{q,p}^{(n)}(T) < \infty\,.$$

Then T is (q,p)-**summing**; the set of these is $\Pi_{q,p}(E,F)$. This is a Banach space.

We write $\pi_{q,p}^{(n)}(E)$ for $\pi_{q,p}^{(n)}(I_E)$ and $\pi_{q,p}(E)$ for $\pi_{q,p}(I_E)$. Also $\pi_p(E)$ for $\pi_{p,p}(E)$, etc.

Theorem Let E be a normed space, and let $n \in \mathbb{N}$. Then

$$\varphi_n^{\max}(E) = \pi_1^{(n)}(E')\,.$$

If $E = F'$, then

$$\varphi_n^{\max}(E) = \pi_1^{(n)}(F)\,.$$

□

The $(p,q)-$multi-norm

Let E be a Banach space, and take p,q with $1 \leq p \leq q < \infty$. Define

$$\|(x_1,\dots,x_n)\|_n^{(p,q)} = \sup\left\{\left(\sum_{j=1}^{n}\left|\langle x_j,\lambda_j\rangle\right|^q\right)^{1/q}\right\}$$

taking the sup over all $\lambda_1,\dots,\lambda_n \in E'$ with $\mu_{p,n}(\lambda_1,\dots,\lambda_n) \leq 1$.

Fact $\{(E^n, \|\cdot\|_n^{(p,q)}) : n \in \mathbb{N}\}$ is a multi-Banach space.

Then $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ is the $(p,q)-$**multi-norm** based on E.

Remarks (1) The $(1,1)$-multi-norm is the maximum multi-norm based on E.

(2) The $(p,q)-$multi-norm over E'', when restricted to E, is the $(p,q)-$multi-norm over E.

(3) The (p,q)-multi-norm induces the norm on $c_0 \otimes E$ given by embedding $c_0 \otimes E$ into $\Pi_{q,p}(E', c_0)$.

The Hilbert multi-norm

Let $H = \ell^2(S)$ be a Hilbert space. For each family $\mathbf{H} = \{H_1, \dots, H_n\}$ of closed subspaces of H such that $H = H_1 \perp \cdots \perp H_n$, set

$$r_{\mathbf{H}}((x_1, \dots, x_n)) = \left(\|P_1 x_1\|^2 + \cdots + \|P_n x_n\|^2\right)^{1/2},$$

where $P_i : H \to H_i$ for $i = 1, \dots, n$ is the projection, and then set

$$\|(x_1, \dots, x_n)\|_n^H = \sup_{\mathbf{H}} r_{\mathbf{H}}((x_1, \dots, x_n)).$$

Then we obtain a multi-norm $(\|\cdot\|_n^H : n \in \mathbb{N})$ based on H. It is the **Hilbert multi-norm**.

Equivalences of multi-norms

Let E be a normed space. Two multi-norms $(\|\cdot\|_n : n \in \mathbb{N})$ and $(|||\cdot|||_n : n \in \mathbb{N})$ based on E are **equivalent** if there are constants C_1 and C_2 such that, for $n \in \mathbb{N}$ and $x \in E^n$, we have

$$C_1 \|x\|_n \leq |||x|||_n \leq C_2 \|x\|_n \, .$$

Theorem [DDPR2] Let H be an infinite-dimensional (complex) Hilbert space. Then:

(i) the Hilbert and $(2, 2)$-multi-norms are equal;

(ii) $\|\cdot\|^H \leq \|\cdot\|^{\max} \leq \frac{2}{\sqrt{\pi}} \|\cdot\|^H$ (and the constant is best-possible);

(iii) they are also equivalent to the (p, p)-multi-norm whenever $p \in [1, 2]$. (But not to any (p, q)-multi-norm when $p < q$.)

(iv) but the (p, p)-multi-norm and the (q, q)-multi-norms are not equivalent whenever $p \neq q$ and $\max\{p, q\} > 2$. □

The standard q-multi-norm on $L^p(\Omega)$

Let Ω be a measure space, and take p, q with $1 \leq p \leq q < \infty$. We consider the Banach space $E = L^p(\Omega)$, with the usual L^p-norm $\|\cdot\|$.

For each family $\mathbf{X} = \{X_1, \ldots, X_n\}$ of pairwise-disjoint measurable subsets of Ω such that $X_1 \cup \cdots \cup X_n = \Omega$, we set

$$r_{\mathbf{X}}((f_1, \ldots, f_n)) = \left(\left\|P_{X_1} f_1\right\|^q + \cdots + \left\|P_{X_n} f_n\right\|^q\right)^{1/q},$$

where $P_X : L^p(\Omega) \to L^p(X)$ is the natural projection.

Finally, $\|(f_1, \ldots, f_n)\|_n^{[q]} = \sup_{\mathbf{X}} r_{\mathbf{X}}((f_1, \ldots, f_n))$.

This is the **standard q-multi-norm.**

Remark Let $q = p$. Then

$$\|(f_1, \ldots, f_n)\|_n^{[q]} = \|\, |f_1| \vee \cdots \vee |f_n| \,\| .$$

Relations between multi-norms

It is a nice game [DDPR2] in using Hölder and summing operators to determine relationships between the above multi-norms. For example:

1) for $1 \leq p \leq q < \infty$, $(\|\cdot\|_n^{[q]}) \leq (\|\cdot\|_n^{(p,q)})$ on $L^1(\Omega)$, with equality for all q when $p = 1$;

2) for $1 < p \leq q < \infty$, $(\|\cdot\|_n^{[q]})$ is not equivalent to $(\|\cdot\|_n^{(p,q)})$ on ℓ^p;

3) for $1 \leq p < q < \infty$, $(\|\cdot\|_n^{(1,q)})$ is equivalent to $(\|\cdot\|_n^{(p,q)})$ on ℓ^1, but not to $(\|\cdot\|_n^{(q,q)})$;

4) Suppose that either $1 \leq p \leq 2 \leq r \leq \infty$ or $1 \leq p < r < 2$. Then the (p,p)-multi-norm is equivalent to the maximum multi-norm on $L^r(\Omega)$, and the stated conditions are exactly best-possible;

5) for $2 < p < q$, the (p,p)-multi-norm is not equivalent to the (q,q)-multi-norm or the maximum multi-norm on any infinite-dimensional Banach space. $\square$

Banach lattice multi-norms

Let $(E, \|\cdot\|)$ be a complex Banach lattice.

Then E is **monotonically bounded** if every increasing net in $E^+_{[1]}$ is bounded above, and **complete** if if every non-empty subset which is bounded above has a supremum.

Examples $L^p(\Omega)$, $L^\infty(\Omega)$, or $C(K)$ with the usual norms and the obvious lattice operations.

Each Banach lattice L^p (for $p \in [1, \infty]$) and $C(K)$ (for K compact) is monotonically bounded, but c_0 is not monotonically bounded.

Each L^p-space is complete, but $C(K)$ is complete iff K is Stonean.

The Banach lattice multi-norm

Definition Let $(E, \|\cdot\|)$ be a Banach lattice. For $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$, set

$$\|(x_1, \ldots, x_n)\|_n^L = \| \, |x_1| \vee \cdots \vee |x_n| \, \| \, .$$

Then $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-Banach space. It is the **Banach lattice multi-norm**.

It generalizes the standard p-multi-norm on $L^p(\Omega)$ and the minimum multi-norms on $L^\infty(\Omega)$ and $C(K)$.

Multi-bounded sets

Let $(E^n, \|\cdot\|_n)$ be a multi-normed space. A subset B of E is **multi-bounded** if

$$c_B := \sup_{n\in\mathbb{N}} \{\|(x_1,\dots,x_n)\|_n : x_1,\dots,x_n \in B\} < \infty\,.$$

For example:

Theorem Let E be a monotonically bounded Banach lattice. Then a subset of E is multi-bounded (for the lattice multi-norm) if and only if it is order-bounded. $\square$

Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be multi-Banach spaces. An operator $T \in \mathcal{B}(E,F)$ is **multi-bounded** if $T(B)$ is multi-bounded in F whenever B is multi-bounded in E. The set of these is a linear subspace $\mathcal{M}(E,F)$ of $\mathcal{B}(E,F)$.

Multi-bounded is the same as 'multi-continuous'.

Multi-bounded operators

For $T \in \mathcal{M}(E,F)$, define

$$\|T\|_{mb} = \sup\{c_{T(B)} : c_B \leq 1\}.$$

Theorem $((\mathcal{M}(E,F), \|\cdot\|_{mb})$ is a Banach space, and $\mathcal{M}(E)$ is a Banach operator algebra. □

[Recall that $\mathcal{M}(E,F)$ depends on the multi-norm structure, and not just on the Banach space, despite the notation.]

In fact, $((\mathcal{M}(E,F)^n, \|\cdot\|_n^{mb}) : n \in \mathbb{N})$ is a multi-Banach space for a suitable multi-norm.

Theorem We can specifically identify the Banach algebra $\mathcal{M}(E)$ in many cases; for example, when E is a complete Banach lattice, it is just the algebra of 'regular' operators. □

(p,q)-multi-bounded subset

Let E be a Banach space, and suppose that $1 \leq p \leq q < \infty$. Consider the (p,q)-multi-norm based on E, and consider the minimum multi-norm based on ℓ^1. Then we write $\mathcal{B}_{p,q}(\ell^1, E)$ for the multi-bounded operators in this case.

Theorem $T \in \mathcal{B}_{p,q}(\ell^1, E)$ if and only if the dual $T' \in \mathcal{B}(E', \ell^\infty)$ is (q,p)-summing. $\square$

The following uses the Pietsch factorization theorem and a theorem of Maurey on the relations of spaces of summing operators.

Theorem Let Ω be a measure space, and take p,q with $1 \leq p \leq q < \infty$. Then every (p,q)-multi-bounded subset of $L^1(\Omega)$ is relatively weakly compact. $\square$

(p,q)-**invariant means**

Let G be a locally compact group, and take p, q with $1 \leq p \leq q < \infty$. A mean $\Lambda \in L^\infty(G)'$ is (p,q)-**invariant** if the set $\{s \cdot \Lambda : s \in G\}$ is (p,q)-multi-bounded. The group G is (p,q)-**amenable** if there is such a mean.

Theorem In fact, G is (p,q)-amenable if and only if it is amenable.

Proof This uses the above weak compactness, the Krein–Smulyan theorem, and the Ryll-Nardzewski fixed point theorem. $\square$

Proof of the conjecture

Theorem Take $p \in (1, \infty)$. Then $L^p(G)$ is injective as a Banach left $L^1(G)$-module if and only if G is amenable.

Proof We modify $J = \mathcal{B}(L^1(G), L^p(G))$ to obtain a Banach left $L^1(G)$-module $\widetilde{J}$, and then, from the definition of injectivity, obtain a morphism R from $\widetilde{J}$ to $L^p(G)$ with suitable properties. We use R to construct a net of continuous linear functionals (Λ_V) on $L^\infty(G)$, and take a weak-$*$ accumulation point Λ. A calculation shows that Λ is a (p,p)-invariant mean, and so, by the earlier theorem, G is amenable. $\square$

The problem of duality

Let E be a Banach space, and let $(\|\cdot\|_n)$ be a multi-norm on $\{E^n : n \in \mathbb{N}\}$.

We might expect that the dual of the multi-normed space is $\mathcal{M}(E, \mathbb{C})$. But this gives just E', and forgets the multi-norm structure.

We could try: $\|\cdot\|'_n$ is the norm on $(E')^n$ which is the dual of the norm $\|\cdot\|_n$ on E^n. But we obtain a dual multi-norm, not a multi-norm, so this fails.

We enter a rather long story to obtain a multi-dual space; we are guided by classical notions of orthogonality.

Hermitian decompositions

Definition Let $(E, \|\cdot\|)$ be a normed space. A direct sum decomposition $E = E_1 \oplus \cdots \oplus E_k$ is **hermitian** if

$$\|\zeta_1 x_1 + \cdots + \zeta_k x_k\| \leq \|x_1 + \cdots + x_k\|$$

for all $\zeta_1, \ldots, \zeta_k \in \overline{\mathbb{D}}$ and $x_1 \in E_1, \ldots, x_k \in E_k$.

The decomposition is hermitian iff the projections are hermitian operators in the sense of numerical range theory.

Fact Let $E = E_1 \oplus \cdots \oplus E_k$ be a hermitian decomposition of a Banach space E. Then

$$E' = E_1' \oplus \cdots \oplus E_k'$$

is a hermitian decomposition of E'.

Small decompositions

Definition Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let $E = E_1 \oplus \cdots \oplus E_k$ be a direct sum decomposition of E. Then:

1) the decomposition is **small** if

$$\|P_1 x_1 + \cdots + P_k x_k\| \leq \|(x_1, \ldots, x_k)\|_k$$

for all $x_1, \ldots, x_k \in E$;

2) the decomposition is **orthogonal** if

$$\|x_1 + \cdots + x_k\| = \|(x_1, \ldots, x_k)\|_k$$

whenever $x_i \in E_i$ for $i = 1, \ldots, k$. (Actually a slightly more complicated definition is required.)

Theorem Small $\Rightarrow$ orthogonal $\Rightarrow$ hermitian. $\square$

Aggravating question Is there an orthogonal decomposition that is not small?

Decompositions and Banach lattices

Let E be a Banach lattice. Recall that the lattice multi-norms are defined by

$$\|(x_1, \ldots, x_n)\|_n^L = \| \, |x_1| \vee \cdots \vee |x_n| \, \|$$

for $x_1, \ldots, x_n \in E$.

Recall also that $E = E_1 \oplus_\perp \cdots \oplus_\perp E_n$ is a **band decomposition** if $|x_i| \wedge \left|x_j\right| = 0$ whenever $x_i \in E_i$, $x_j \in E_j$, and $i \neq j$.

Easy: a band decomposition is small, and hence orthogonal, for the lattice multi-norm. What about the converse?

Kalton's theorem

Let $E = E_1 \oplus \cdots \oplus E_k$ be orthogonal with respect to the lattice multi-norm. Then

$$\| \, |x_1| \vee \cdots \vee |x_k| \, \| = \|x_1 + \cdots + x_k\|$$

whenever $x_1 \in E_1, \ldots, x_k \in E_k$.

Theorem - Nigel Kalton Ka This already implies that the decomposition is a band decomposition (for complex Banach lattices). $\square$

[In fact Kalton shows that the complex linear span of the hermitian operators on a Banach lattice is a C^*-algebra.]

Theorem Let E be a (complex) Banach lattice. Then the following properties of a decomposition are equivalent:

(a) it is a band decomposition;

(b) it is orthogonal for the lattice multi-norm;

(c) it is small for the lattice multi-norm. $\square$

Orthogonality - Examples

(1) Let H be a Hilbert space with the Hilbert multi-norms. Then decompositions are orthogonal/small/hermitian if and only if they are orthogonal in the classical sense.

(2) Let $E = \ell^p(S)$ have the standard p-multi-norm. Then a decomposition $E = E_1 \oplus \cdots \oplus E_k$ is orthogonal/small/hermitian if and only if there is a partition $\{S_1, \ldots, S_k\}$ of S such that $E_j = \ell^p(S_j)$ $(j = 1, \ldots, k)$.

(3) Let $E = \ell^p(S)$ have the standard (p, q)-multi-norm, with $q \neq p$. Then there are no non-trivial orthogonal decompositions of E.

(4) Take K compact, and let $C(K)$ have any multi-norm. Then a decomposition $C(K) = E_1 \oplus \cdots \oplus E_k$ is orthogonal/small/hermitian if and only if there is a partition $\{K_1, \ldots, K_k\}$ of K into clopen sets such that $E_j = C(K_j)$ $(j = 1, \ldots, k)$. □

Families of decompositions

Definition Let $(E, \|\cdot\|)$ be a normed space, and consider a family $\mathcal{K}$ of direct sum decompositions of E. Then $\mathcal{K}$ is **closed** provided that the following conditions are satisfied:

(C1) $E_{\sigma(1)} \oplus \cdots \oplus E_{\sigma(n)} \in \mathcal{K}$ whenever $n \in \mathbb{N}$, $\sigma \in \mathfrak{S}_n$, and $E_1 \oplus \cdots \oplus E_n \in \mathcal{K}$;

(C2) $F \oplus E_3 \oplus \cdots \oplus E_n \in \mathcal{K}$ whenever $n \geq 3$, $E_1 \oplus \cdots \oplus E_n \in \mathcal{K}$, and $F = E_1 \oplus E_2$;

(C3) $\mathcal{K}$ contains the family of all trivial direct sum decompositions.

The families $\mathcal{K}_{\text{herm}}$ of all hermitian decompositions, $\mathcal{K}_{\text{small}}$ of all small decompositions, and $\mathcal{K}_{\text{orth}}$ of all orthogonal decompositions are closed families of decompositions.

Multi-norms from families of decompositions

Let $(E, \|\cdot\|)$ be a normed space, and consider a closed family $\mathcal{K}$ of hermitian decompositions of E. For $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$, set

$$\|(x_1, \ldots, x_n)\|_n^{\mathcal{K}} = \sup \{\|P_1 x_1 + \cdots + P_n x_n\|\} .$$

Then $((E^n, \|\cdot\|_n^{\mathcal{K}}) : n \in \mathbb{N})$ is a multi-normed space, and each member of $\mathcal{K}$ is a small decomposition of E with respect to this multi-norm. This is the multi-norm **generated** by $\mathcal{K}$.

For example, the family of trivial decompositions generates the minimum multi-norm.

Duals of hermitian decompositions

Consider a closed family $\mathcal{K}$ of hermitian decompositions of E. The dual family is

$$\mathcal{K}' = \{E_1' \oplus \cdots \oplus E_k' : E_1 \oplus \cdots \oplus E_k \in \mathcal{K}\} ,$$

and it generates a multi-norm $(\|\cdot\|_{n,\mathcal{K}}^{\dagger} : n \in \mathbb{N})$ on the family $\{(E')^n : n \in \mathbb{N}\}$.

Dual multi-norms

Definition Let $(E, \|\cdot\|)$ be a normed space, and let $\mathcal{K}$ be a closed family of hermitian decompositions of E. Then the multi-norm on $\{(E')^n : n \in \mathbb{N}\}$ generated by $\mathcal{K}'$ is denoted by

$$(\|\cdot\|_{n,\mathcal{K}}^{\dagger} : n \in \mathbb{N})\,.$$

The multi-normed space

$$(((E')^n, \|\cdot\|_{n,\mathcal{K}}^{\dagger}) : n \in \mathbb{N})$$

is the **multi-dual space** with respect to $\mathcal{K}$.

Theorem For complete Banach lattices, the family $\mathcal{K}$ of band decompositions (so that $\mathcal{K} = \mathcal{K}_{\mathsf{small}} = \mathcal{K}_{\mathsf{orth}}$) generates the lattice multi-norm, and the family $\mathcal{K}'$ generates the lattice multi-norm on E'. □

We have **multi-biduals**, **multi-reflexive spaces**, and quite a few examples,... and queries.

A consequence of duality

We have the following, which was one point of the definitions.

Theorem For $1 < p < \infty$, let the families $\{(\ell^p)^n : n \in \mathbb{N}\}$ have the standard p-multi-norm. Then the multi-dual of the multi-normed space $\{(\ell^p)^n : n \in \mathbb{N}\}$ is $\{(\ell^q)^n : n \in \mathbb{N}\}$ with the standard q-multi-norm, where $q = p'$. Hence these multi-normed spaces are multi-reflexive. $\square$

Starting from a multi-norm

We start from a multi-normed space $(E^n, \|\cdot\|_n)$. The theory works if the multi-norm is **orthogonal** in the sense that

$$\|(x_1, \ldots, x_n)\|_n = \sup\{\|P_1x_1 + \cdots + P_nx_n\|\}\ ,$$

for all $x_1, \ldots, x_n \in E$, taking the supremum over all small decompositions of E. Then we have a multi-dual space.

For example, what properties of a normed space E and/or p, q mean that the (p, q)-multi-norm on E is orthogonal - and, when it is, what is the multi-dual?

And what does this say for the embryonic theory of multi-Banach algebras?

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