Abstract Harmonic Analysis & Quantum Information Theory

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Outline

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Quantum States

Definition

Let H be a Hilbert space. A quantum state on H is a positive trace-class operator $\rho \in \mathcal{T}(H)$ of trace 1. We denote the set of states on H by $\mathcal{D}(H)$.

Examples:

- Qubits: $\mathcal{D}(\mathbb{C}^2)$, e.g., spin-1/2 particle, photon.
- n-qubits - $\mathcal{D}((\mathbb{C}^2)^{\otimes^n})$.
- Hydrogen orbitals - $\mathcal{D}(L^2(\mathbb{R}^3))$.

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Alternative definition: A normal UCP map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$.

Quantum Channels

Theorem (Haagerup '80; Blecher-Smith '92)

$\Phi \in \mathcal{CB}^\sigma(\mathcal{B}(H))$ is CP if and only if $\exists$ a net $(a_i)_{i\in I}$ in $\mathcal{B}(H)$ such that

$$\Phi(x) = w^* - \sum_{i\in I} a_i x a_i^*, \qquad x \in \mathcal{B}(H).$$

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Terminology:

- Φ is trace preserving (TP) if $w^* - \sum_{i\in I} a_i^* a_i = I$.
- Φ is bistochastic (BS) if it is UCP & TP.
- $(a_i)_{i\in I}$ are the Kraus operators of Φ.

Outline

1. Quantum Channels
2. Quantum Group Channels
3. Examples & Applications
4. Kac-Paljutkin Algebra

Locally Compact Quantum Groups

Definition (Kustermans-Vaes '00)

A LCQG $\mathbb{G} = (M, \Gamma, \varphi, \psi)$

- M is a von Neumann algebra
- $\Gamma : M \to M \bar{\otimes} M$ is a co-multiplication: normal, unital, isometric $*$-homomorphism, co-associative

$$(\Gamma \otimes \iota) \circ \Gamma = (\iota \otimes \Gamma) \circ \Gamma$$

- φ is a *left Haar weight* on M:

$$\varphi((\omega \otimes \iota)\Gamma(x)) = \omega(1)\varphi(x), \quad x \in \mathcal{M}_\varphi, \omega \in M_*$$

- ψ is a *right Haar weight* on M:

$$\psi((\iota \otimes \omega)\Gamma(x)) = \omega(1)\psi(x), \quad x \in \mathcal{M}_\psi, \omega \in M_*$$

Duality & Examples

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Kac-Paljutkin Algebra: $\mathbb{G}_8 = (VN(G_8), \Gamma_\Omega, \varphi)$ where

- $G_8 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\alpha \mathbb{Z}_2$, α is permutation
- $\Gamma_\Omega(x) = \Omega \Gamma_s(x) \Omega^*$
- Ω is a unitary in $VN(G_8) \otimes VN(G_8)$.

Universal C^*-Algebra

Notation: $L^\infty(\mathbb{G}) := M$, $L^1(\mathbb{G}) := M_*$, $L^2(\mathbb{G}) := L^2(M, \varphi)$.

Theorem (Kustermans '01)

For every $\mathbb{G}$ there exists a universal C^*-algebra $C_u(\mathbb{G})$ such that $C_u(\mathbb{G})^*$ is a Banach algebra containing $L^1(\mathbb{G})$ as a closed ideal.

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Definition

A positive definite functional on $\mathbb{G}$ is an element of $(C_u(\mathbb{G})^*)^+$.

- $\mathcal{P}(\mathbb{G}_a) = M^+(G)$ and $\mathcal{P}(\mathbb{G}_s) = \mathcal{P}(G)$.

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- $\mathcal{P}(\mathbb{G}_a) = M^+(G)$ and $\mathcal{P}(\mathbb{G}_s) = \mathcal{P}(G)$.
- $\mu \in C_u(\mathbb{G})^* \rightsquigarrow m_\mu \in \mathcal{CB}(L^1(\mathbb{G})) \rightsquigarrow$ $(\tilde{m}_\mu)^* = \Theta(\mu) \in \mathcal{CB}^\sigma(\mathcal{B}(L^2(\mathbb{G})))$ (J-N-R '09)

Quantum Group Channels

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If $\mu \in \mathcal{P}_1(\mathbb{G})$ *then* $\Theta(\mu)$ *is UCP, i.e., a quantum channel.*

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Quantum HA: 1-1 correspondence between $\mathcal{P}(\mathbb{G})$ and unitary co-reps of $\hat{\mathbb{G}}$ (*Kustermans* '01).

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Theorem (C)

If $\mu = (\pi, \xi) \in \mathcal{P}_1(\mathbb{G})$ is given by U_μ, then

$$\Theta(\mu)(x) = w^* - \sum_{i \in I} U_i^* x U_i, \quad x \in \mathcal{B}(L^2(\mathbb{G}))$$

where $U_i = (\omega_{\xi, e_i} \otimes \iota)(U_\mu)$, $(e_i)_{i \in I}$ o.n. basis for H_μ.

Outline

$G = \mathbb{Z}_2$

- $L^\infty(\mathbb{Z}_2)$: Bit-flip channel

$$\Theta(\mu)(\rho) = \mu(0)\rho + \mu(1)X\rho X$$

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- $VN(\mathbb{Z}_2)$: Phase-flip channel

$$\hat{\Theta}(\varphi)(\rho) = p\rho + (1-p)Z\rho Z$$

$Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = M_{\chi^{(1)}},\ \varphi = p\chi^{(0)} + (1-p)\chi^{(1)}.$

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Dual Channels

$|G| < \infty$

- $\forall \ \mu \in M_1^+(G)$

$$\Theta(\mu)(\rho) = \sum_{s \in G} \mu(s) \lambda(s^{-1}) \rho \lambda(s)$$

is random unitary $\Rightarrow$ BS quantum channel.

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- Fixed points $\mathcal{H}_{\Theta(\mu)} = (\mathcal{H}_\mu \cup \mathcal{R}(G))''$ always contain (*K-N-R* '11)

$$\mathcal{R}(G) \cong \bigoplus_{[\pi]\in\hat{G}} I_{d_\pi} \otimes M_{d_\pi}(\mathbb{C})$$

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- Noiseless Subsystems method of quantum error correction.

$|G| < \infty$

- $\mathcal{P}_1(G) \ni \varphi(s) = \langle \pi(s)\xi, \xi \rangle$. If $\langle (e_i)_{i=1}^{d_\pi} \rangle = H_\pi$, then

$$\hat{\Theta}(\varphi)(\rho) = \sum_{i=1}^{d_\pi} M_{\xi_i} \rho M_{\xi_i}^*$$

is a BS quantum channel where $\xi_i(s) = \langle e_i, \pi(s)\xi \rangle$.

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- Dual to random unitaries $\Theta(\mu)$
- Question: Which $\hat{\Theta}(\varphi)$ are outside $\mathcal{RU}_{|G|}$?

Asymptotic Quantum Birkoff Conjecture

Theorem (Birkoff)

The set of $d \times d$ bistochasitc matrices is a convex set whose extreme points are the $d!$ permutation matrices.

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AQBC

Any BS channel $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ satisfies

$$\lim_{n \to \infty} \|\Phi^{\otimes^n} - \mathcal{RU}(M_d(\mathbb{C})^{\otimes^n})\|_{cb} = 0.$$

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- Recently proved false (*Haagerup-Musat* '11, *Shor-Oza-Ostrev* '11)

Maximally Extreme Positive Definite Functions

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Theorem (Haagerup-Musat '11; C)

If $\varphi \in \mathcal{P}_1(G)$ is maximally extreme with $d_\pi \geq 2$, then $\hat{\Theta}(\varphi)$ does not satisfy AQBC.

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Question: When is φ maximally extreme?

Geometric Interpretation

$\pi : \mathcal{G} \to \mathcal{B}(\mathbb{C}^2)$ acts by conjugation on the Bloch Sphere:

$$\mathcal{D}(\mathbb{C}^2) \ni \rho = \frac{1}{2}(I + \overrightarrow{r_\rho} \cdot \overrightarrow{\sigma})$$

where $\overrightarrow{\sigma} = (X, Y, Z)$ - Pauli matrices, $\overrightarrow{r_\rho} \in \mathbb{R}^3$, $\|\overrightarrow{r_\rho}\| \leq 1$.

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Theorem (Helm-Strunz '09; C)

$\varphi = (\pi, \xi) \in \mathcal{P}_1(G)$ *is maximally extreme* $\Leftrightarrow$ $Aff(\overrightarrow{r_g})_{g \in G} = \mathbb{R}^3$, *i.e.,* $vol(cov(\overrightarrow{r_g})_{g \in G}) > 0$.

$G = S_3$

$\pi : S_3 \to \mathcal{B}(\mathbb{C}^2)$ 2d irred rep

$$\xi = \frac{1}{\sqrt{5}}(-1, 2i)$$

$\varphi = (\pi, \xi)$ is

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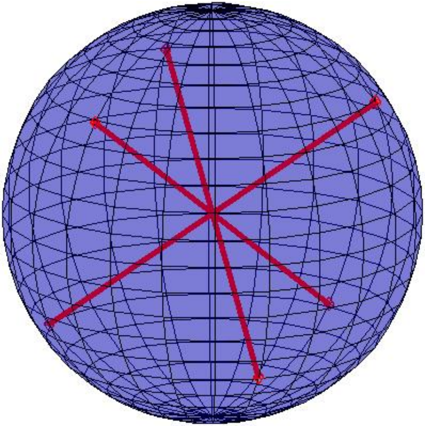

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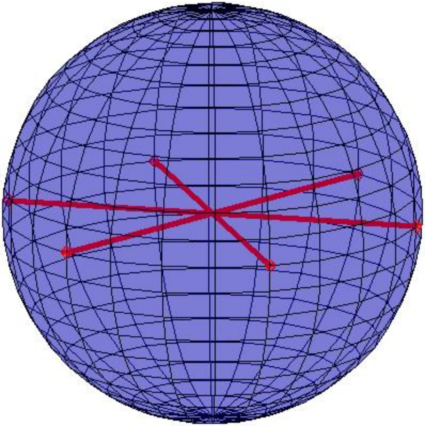

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$L^1(\mathbb{G}_8) = (A(G_8), \star) \Rightarrow \mathcal{P}_1(\mathbb{G}_8) = \mathcal{P}_1(G_8)$,

$$\Theta(\mu)(x) = \sum_{i=1}^{8} W_i^* x W_i, \quad x \in \mathcal{B}(L^2(G_8))$$

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$$\mathcal{H}_{\Theta(\mu)} = (\mathcal{H}_\mu \cup L^\infty(\hat{\mathbb{G}}_8)')'' \quad (K\text{-}N\text{-}R \text{ '11})$$

Idempotent States in $L^1(\mathbb{G}_8)$

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Solutions: $G_8 = \{e, s, a, as, b, bs, ab, abs\}$ - characteristic functions of

$$G_1 = \{e\}, \quad G_2 = \{e, b\}, \quad G_3 = \{e, ab\},$$
$$G_4 = \{e, ab, as, bs\}, \quad G_5 = \{e, s, ab, abs\},$$
$$G_6 = \{e, a\}, \quad G_7 = \{e, a, b, ab\}, \quad G_8.$$

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$$\mathcal{H}_{\chi_4} = \text{span}\{\lambda(e), \lambda(ab), -2i\lambda(as) - \lambda(s) + \lambda(abs), \lambda(as) + \lambda(bs)\}$$
$$\mathcal{H}_{\chi_5} = \text{span}\{\lambda(e), \lambda(ab), -2i\lambda(s) - \lambda(as) + \lambda(bs), \lambda(s) + \lambda(abs)\}$$

Thank You!

Convolution in $L^1(\mathbb{G}_8)$:

$$\mu \star_\Omega \nu(r) = \sum_{g,h \in \mathbb{Z}_2^2} \sum_{s,t \in \mathbb{Z}_2^2} c(g,h)\overline{c(s,t)}\mu((g,e)r(s,e))\nu((h,e)r(t,e))$$

Fundamental Unitary:

$$W = \sum_{r \in G_8} \sum_{x,y,s,t \in \mathbb{Z}_2^2} \overline{c(x,y)}c(s,t)\lambda(r)^*\lambda(\alpha_r(s),e)\lambda(x,e) \otimes M_{\delta_r}\lambda(\alpha_r(t),e)\lambda$$