Real Banach algebras as $C(\mathcal{K})$-algebras

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We will give a proof using only real methods (i.e. a proof not involving complex function theory) of the following theorem.

Theorem *Let A be a commutative unital real Banach algebra satisfying*

$$(*) \quad r(a^2) \leq r(a^2 + b^2) \text{ for all } a, b \in A,$$

where r denotes the spectral radius, $r(a) = \lim \|a^n\|^{1/n}$. Then there exists a compact Hausdorff space $\mathcal{K}$ and a homomorphism $\hat{}$ of A into $C(\mathcal{K})$, the space of all real-valued continuous functions on $\mathcal{K}$, such that $r(a) = \sup \|\hat{a}_{\infty,\mathcal{K}}\|$.

The main tool in the proof is the set A_+ of squares in A, and its closure $\mathrm{cl}(A_+)$. Clearly, condition $(*)$ extends to $\mathrm{cl}(A_+)$. We frequently use the square root lemma, SQRT, which says that if $\|a\| \leq 1$ or if $r(a) < 1$ then $e - a \in A_+$ where e is the unit in A. (See f.ex. T. W. Palmer's book on Banach algebras.)

Lemma 1 *Let φ in the dual space A^* satisfy $\varphi(a^2) \geq 0$ for all a in A. Then $\|\varphi\| = \varphi(e)$.*

Proof Let $\|u\| \leq 1$. Then $e - u \in A_+$ so that $\varphi(e - u) \geq 0$ and thus $\varphi(u) \leq \varphi(e)$.

In general A_+ or $\mathrm{cl}(A_+)$ need not be convex, see [2]. However, if $(*)$ holds then we have,

Lemma 2 $\mathrm{cl}(A_+) = \bigcup_{t>0} t(e - U)$, *where U is the r-unit ball, $U = \{u : r(u) \leq 1\}$. In particular $\mathrm{cl}(A_+)$ is convex.*

Proof Suppose a is in $\mathrm{cl}(A_+)$ and $\|a\| < 1$. Then $a = e - (e - a)$ where $r(e - a) \leq r((e - a) + a) = 1$. Suppose conversely that $a = e - u$, where $r(u) \leq 1$. Then $a = \lim_{t \to 1^-} e - tu \in \mathrm{cl}(A_+)$.

Let $\mathcal{S}_+$ denote the set

$$\mathcal{S}_+ = \{\varphi \in A^* : \|\varphi\| = 1 \text{ and } \varphi \geq 0 \text{ on } A_+\},$$

a weak*-closed convex subset of A^*.

Lemma 3 *Suppose $\|a^2\| > 1$. Then there is φ in $\mathcal{S}_+$ satisfying $\varphi(a) > 1$.*

Proof Note that a^2 is not in the norm closed convex set $e - \mathrm{cl}(A_+)$ because if $a^2 = e - b$ with b in $\mathrm{cl}(A_+)$ then $r(a^2) = r(e - b) \leq r((e - b) + b) = 1$.

Let φ be an element of A^* of norm 1, separating a^2 from $e - \mathrm{cl}(A_+)$,

$$\varphi(a^2) > \varphi(e - b) \text{ for all } b \in \mathrm{cl}(A_+).$$

Then, since $\mathrm{cl}(A_+)$ is closed under multiplication by positive numbers, φ must be positive on A_+ and is thus in $\mathcal{S}_+$.

Lemma 4 *Let φ be an element of $\mathcal{S}_+$. Then φ is multiplicative.*

Proof For b in A let φ_b denote the functional $\varphi_b(c) = \varphi(bc)$. By Lemma 1, $\|\varphi_{a^2}\| = \varphi_{a^2}(e) = \varphi(a^2)$ for a in A. Thus $\varphi(a^2c) = 0 = \varphi(a^2)\varphi(c)$ for all c if $\varphi(a^2) = 0$. Suppose $\|a^2\|<1$ and $\varphi(a^2) > 0$. Since

$$\varphi = \varphi_{a^2} + \varphi_{e-a^2}$$

and since φ_{a^2} and φ_{e-a^2} are both non-negative on A_+, using Lemma 1, we deduce that $\varphi = \lambda\varphi_{a^2}$, where $\lambda = 1/\varphi_{(a^2)}$. Thus $\varphi(a^2c) = \varphi(a^2)\varphi(c)$ for all c in A. Since every element of A is the difference of two squares we are done.

Proof of Theorem Let X denote the weak*-closure of the set of extreme points of $\mathcal{S}_+$ equipped with the weak*-topology and map A into $C(X)$ via $a \to \hat{a}$, where $\hat{a}(\varphi) = \varphi(a)$. This map is a homomorphism by Lemma 4. Since each element of X is a homomorphism of A into the reals, $\|\hat{a}\|_{\infty,X} \leq r(a)$. Since $r(a^2) = r(a)^2$, the reverse inequality follows from Lemma 3.

[1] Albiac, F. and Briem, E. *$C(\mathcal{K})$-representations of real Banach algebras*, J. Aust. Math. Soc. 88 (2010), 289-300.

[2] Albiac, F. and Briem, E. *Real Banach algebras $C(\mathcal{K})$-algebras*, to appear in Quart. J. Math. Advance Access published April 7, 2011