Approximate Identities in Convolution Algebras of some Free Quantum Groups

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Motivation: Approximate identities in $A(\mathbb{F}_N)$.

Consider $\mathbb{F}_N$, the free group on $N \geq 2$ generators, with group von Neumann algebra $VN(\mathbb{F}_N) = \lambda(\mathbb{F}_N)'' \subset \mathcal{B}(\ell^2(\mathbb{F}_N))$, and Fourier algebra

$$A(\mathbb{F}_N) = VN(\mathbb{F}_N)_* = \{g \mapsto \langle \lambda(g)\xi|\eta\rangle : \xi, \eta \in \ell^2(\mathbb{F}_N)\}.$$

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Yes!: Let $MA(\mathbb{F}_N) =$ multiplier algebra of $A(\mathbb{F}_N)$
$= \{\varphi \in \ell^\infty(\mathbb{F}_N) : \varphi A(\mathbb{F}_N) \subseteq A(\mathbb{F}_N)\}$.
Then $A(\mathbb{F}_N) \hookrightarrow MA(\mathbb{F}_N)$, and $\|u\|_{MA(\mathbb{F}_N)} \leq \|u\|_{A(\mathbb{F}_N)}$ $(u \in A(\mathbb{F}_N))$.

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Theorem (Haagerup, 1979)

$A(\mathbb{F}_N)$ has a (finitely supported) approximate identity $\{e_\alpha\}_{\alpha \in S}$ such that $\|e_\alpha\|_{MA(\mathbb{F}_N)} = 1$ $(\forall \alpha \in S)$.

Better yet: $\mathbb{F}_N$ is 1-weakly amenable **[de Canniere & Haagerup, 1985]**. I.e., can take e_α s.t. $\|e_\alpha\|_{M_{cb}A(\mathbb{F}_N)} = 1$ $\forall \alpha \in S$.

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$\exists$ surjective $*$-homomorphisms

$$C_u(U_N^+) \twoheadrightarrow C(U_N), C^*(\mathbb{F}_N), \qquad C_u(O_N^+) \twoheadrightarrow C(O_N), C^*((\mathbb{Z}/2\mathbb{Z})^{*N}).$$

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$C_u(U_N^+)$ and $C_u(O_N^+)$ are C*-bialgebras with coproduct

$$\Delta(u_{ij}) = \sum_{r=1}^{N} u_{ir} \otimes u_{rj}, \qquad \Big(\underbrace{(\mathrm{id}\otimes\Delta)\circ\Delta = (\Delta\otimes\mathrm{id})\circ\Delta}_{\text{coassociative}}\Big).$$

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Bialgebra structure $\implies$ get compact quantum groups.

Definition (S. Wang, 1993)

$U_N^+ := (C_u(U_N^+), \Delta)$ is the free UNITARY quantum group.
$O_N^+ := (C_u(O_N^+), \Delta)$ is the free ORTHOGONAL quantum group.

The reduced versions of O_N^+, U_N^+.

For $\mathbb{G} = U_N^+, O_N^+$, $\exists!$ Haar state $h : C_u(\mathbb{G}) \to \mathbb{C}$, s.t.

$$(h \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes h)\Delta = h(\cdot)1_{C_u(\mathbb{G})}.$$

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GNS construction: Put $L^2(\mathbb{G}) := L^2(C_u(\mathbb{G}), h)$ and let $\lambda : C_u(\mathbb{G}) \to \mathcal{B}(L^2(\mathbb{G}))$ be the left regular (GNS) representation associated to h.

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Get the reduced C*-algebra of $\mathbb{G}$: $C_r(\mathbb{G}) = \lambda(C_u(\mathbb{G})) \subseteq \mathcal{B}(L^2(\mathbb{G}))$, and the reduced von Neumann algebra of $\mathbb{G}$: $L^\infty(\mathbb{G}) := C_r(\mathbb{G})''$.

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Δ drops to a (normal, faithful) coproduct $\Delta_r : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$, making $(L^\infty(\mathbb{G}), \Delta_r, h)$ a von Neumann algebraic CQG.

Just like $A(G)$, $L^1(\mathbb{G}) := L^\infty(\mathbb{G})_*$ is a CC Banach algebra with multiplication $(\Delta_r)_*$.

The structure of $C_r(\mathbb{G})$, $L^\infty(\mathbb{G})$.

For $\mathbb{G} = O_N^+, U_N^+$, $C_r(\mathbb{G})$ and $L^\infty(\mathbb{G})$ have a lot in common with $C_r^*(\mathbb{F}_N)$ and $VN(\mathbb{F}_N)$:

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- For $N \geq 3$, $L^\infty(\mathbb{G})$ is a non-injective, solid II_1-factor. $C_r(\mathbb{G})$ is non-nuclear, simple, exact, projectionless. [Banica, Vaes, Vergnioux, Voigt]

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Questions

(Hard): Is $L^\infty(O_N^+), L^\infty(U_N^+) \cong VN(\mathbb{F}_k)$ for other values of $N \geq 3$, $k \geq 2$?

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(Easier/Related): Can we find other properties of $L^\infty(\mathbb{G}), C_r(\mathbb{G}), L^1(\mathbb{G})$ shared/not shared with $VN(\mathbb{F}_k)$, $C_r^*(\mathbb{F}_k)$, $A(\mathbb{F}_k)$?

Approximate units in $L^1(\mathbb{G})$ and approximation properties for $L^\infty(\mathbb{G})$, $C_r(\mathbb{G})$.

Recall that for $N \geq 3$, $L^\infty(\mathbb{G})$ is non-injective $\iff$ $\mathbb{G}$ is not co-amenable $\iff$ $L^1(\mathbb{G})$ does NOT have a bounded approximate identity.

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However, by analogy with $A(\mathbb{F}_N)$, consider the (left) multiplier algebra

$$M^\ell(L^1(\mathbb{G})) = \{L \in \mathcal{B}(L^1(\mathbb{G})) \mid L(\omega_1 * \omega_2) = (L\omega_1) * \omega_2 \ \forall \omega_1, \omega_2 \in L^1(\mathbb{G})\}.$$

Then again, $L^1(\mathbb{G}) \hookrightarrow M^\ell(L^1(\mathbb{G}))$ is a contraction.

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Theorem (B.)

$L^1(\mathbb{G})$ has a central approximate identity $\{e_\alpha\} \subset ZL^1(\mathbb{G})$ s.t. $\|e_\alpha\|_{M^\ell L^1(\mathbb{G})} = \|e_\alpha\|_{M^r L^1(\mathbb{G})} = 1 \ \forall \alpha$.

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From our proof of this result, we also answer some questions of Vaes and Vergnioux concerning the structure of $L^\infty(\mathbb{G})$, $C_r(\mathbb{G})$:

Theorem (B.)

$L^\infty(O_N^+)$ and $L^\infty(U_N^+)$ have the Haagerup approximation property (HAP).
$C_r(O_N^+)$ and $C_r(U_N^+)$ have the metric approximation property (MAP).

About the proof.

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(1). For each $r \in (0,1)$, the radial function $g \mapsto \varphi_r(g) = r^{\ell(g)}$ is positive definite $\iff$ the maps $M_{\varphi_r}\lambda(g) = \varphi_r(g)\lambda(g)$ are (normal) U.C.P. radial multipliers of $VN(\mathbb{F}_N)$, and $\lim_{r\to 1} M_{\varphi_r} = \mathrm{id}_{VN(\mathbb{F}_N)}$ point σ-weakly.

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$\implies$ Finitely supported truncations of the φ_r's produce multipliers of $VN(\mathbb{F}_N)$ still converging to $\mathrm{id}_{VN(\mathbb{F}_N)}$, with norm-control. By duality, the truncated φ_r's yield the desired approximate identity for $A(\mathbb{F}_N)$.

For O_N^+.

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Peter-Weyl Theory: $L^2(O_N^+) = \bigoplus_{k\in\mathbb{N}_0} L^2_{(k)}(O_N^+)$, where $L^2_{(k)}(O_N^+) = \Lambda_h(\text{span}\{v_{ij}^{(k)} : 1 \le i, j \le d_k\})$. - coeff. space of $V^{(k)}$.

Can define a length function

$$\ell : \widehat{O}_N^+ \cong \mathbb{N}_0 \to [0, \infty), \qquad \ell(k) = \min\{r \in \mathbb{N}_0 \mid V^{(k)} \subset U^{\boxtimes r}\} = k.$$

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$\widehat{O}_N^+$ has Property RD w.r.t. ℓ. I.e., the orthogonal projections

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Thus, if we could find a net of U.C.P. multipliers $\{L_\alpha\} \subset M^\ell(L^1(O_N^+))$, with "very rapid decay" w.r.t. ℓ, and with $(L_\alpha)^* \to \mathrm{id}$ ptwse. σ-weakly, then we could use Property RD to truncate the L_α's to get a $M^\ell(L^1(O_N^+))$-BAI for $L^1(O_N^+)$.

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Definition

A radial multiplier of $L^\infty(O_N^+)$ is an operator

$$T = \sum_{k \in \mathbb{N}_0} \alpha_k p_k \in \mathcal{B}(L^2(O_N^+))$$

such that $T|_{L^\infty(O_N^+)} = L^*$ for some $L \in M^\ell(L^1(O_N^+))$.

By analogy with $\mathbb{F}_N$, we may hope that $T_r = \sum_{k \in \mathbb{N}_0} r^k p_k \in \mathcal{B}(L^2(O_N^+))$is a U.C.P. radial multiplier for $0 < r < 1$.

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Let $S_k(x) = k$'th Chebychev *II* polynomial: $S_0(x) = 1$, $S_1(x) = x$, and $xS_k(x) = S_{k-1}(x) + S_{k+1}(x)$ $(k \geq 2)$.

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For $t \in (2, N)$,

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Observe: Can find $D > 0$ s.t. $0 < \frac{S_k(t)}{S_k(N)} \leq D\left(\frac{t}{N}\right)^k$ - exponential decay!, while $\lim_{t\to N} T_t|_{L^\infty(O_N^+)} = \mathrm{id}_{L^\infty(O_N^+)}$ pointwise σ-weakly. (yields HAP).

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But why is T_t a U.C.P. radial multiplier?

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There is a bijection between states $\varphi \in \mathcal{S}\big(C[-N, N]\big)$ *and U.C.P. radial multipliers given by* $\varphi \longleftrightarrow T_\varphi = \sum_{k\in\mathbb{N}_0} \alpha_k^\varphi p_k$ *given by* $\alpha_k^\varphi = \frac{\varphi(S_k)}{S_k(N)}$.

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($\Longrightarrow$): Uses M. Daws' characterisation of C.P. multipliers of QGs. □

Work in progress.

What is the Cowling-Haagerup constant $\Lambda_{cb} \in [1, \infty]$ for $L^\infty(O_N^+)$? (Is $\widehat{O}_N^+$ weakly amenable with constant Λ_{cb}?)

More on averaging.

Since the Haar state h is a faithful normal trace on $L^\infty(O_N^+)$, and

$$\Delta_r : L^\infty(O_N^+) \to L^\infty(O_N^+)\overline{\otimes}L^\infty(O_N^+)$$

is an injective normal $*$-homomorphism, $\exists!$ faithful normal $h \otimes h$-preserving conditional expectation

$$E : L^\infty(O_N^+)\overline{\otimes}L^\infty(O_N^+) \to \Delta_r(L^\infty(O_N^+)).$$

We now average L_ψ^* with respect to E:

$$L_\psi^* \mapsto \Delta_r^{-1} \circ E \circ (\kappa \circ L_\psi^* \circ \kappa \otimes \mathrm{id}) \circ \Delta_r \in \mathcal{CP}_\sigma(L^\infty(O_N^+)),$$

where $\kappa \in \mathcal{B}(L^\infty(O_N^+))$ is the antipode. (A bdd. $*$-automorphism).

One can write down concrete formulae for E, κ, L_φ^* etc. and check that the RHS actually equals T_φ!!!