Applications of a new kind of positivity in operator algebras

David Blecher and Charles Read
University of Houston and the University of Leeds

David Blecher and Matthew Neal
University of Houston and Denison University

Advertisement: AMS/SAMS meeting and follow-on conference

"Operator and Banach algebras, and non-commutative analysis"

29 Nov to 3 Dec, 2011: AMS and the South African Math Society joint congress, in the coastal town of Port Elizabeth, South Africa. The conference website is

http://www.nmmu.ac.za/sams-ams2011

David Blecher (Houston), Garth Dales (Leeds, UK), Jan Fourie and Louis E Labuschagne (Potchefstroom), and Anton Stroh (Pretoria)

Followed with a 3-day conference on similar topics on 5/6/7 December 2011, at the University of Pretoria.

Email me if interested (dblecher@math.uh.edu)

Advertisement: AMS/SAMS meeting and follow-on conference

"Operator and Banach algebras, and non-commutative analysis"

29 Nov to 3 Dec, 2011: AMS and the South African Math Society joint congress, in the coastal town of Port Elizabeth, South Africa. The conference website is

http://www.nmmu.ac.za/sams-ams2011

David Blecher (Houston), Garth Dales (Leeds, UK), Jan Fourie and Louis E Labuschagne (Potchefstroom), and Anton Stroh (Pretoria)

Followed with a 3-day conference on similar topics on 5/6/7 December 2011, at the University of Pretoria.

Email me if interested (dblecher@math.uh.edu)

(GPOTS 2012: Houston)

Abstract

We use a substitute for the positive cone in a C^*-algebra, to uncover much hidden positivity in general operator algebras

This gives a device/strategy to generalize results hitherto available only for C^*-algebras. In particular, to generalize C^*-algebraic results relying on positivity, and in particular on the existence of a positive cai.

Abstract

We use a substitute for the positive cone in a C^*-algebra, to uncover much hidden positivity in general operator algebras.

This gives a device/strategy to generalize results hitherto available only for C^*-algebras. In particular, to generalize C^*-algebraic results relying on positivity, and in particular on the existence of a positive cai.

We give several examples of this. For example, in the first part (joint work with C. Read) we dissolve the main mysteries surrounding contractive approximate identities (cai's) in operator algebras and in their one-sided ideals, by showing how they all arise.

Abstract

We use a substitute for the positive cone in a C^*-algebra, to uncover much hidden positivity in general operator algebras.

This gives a device/strategy to generalize results hitherto available only for C^*-algebras. In particular, to generalize C^*-algebraic results relying on positivity, and in particular on the existence of a positive cai.

We give several examples of this. For example, in the first part (joint work with C. Read) we dissolve the main mysteries surrounding contractive approximate identities (cai's) in operator algebras and in their one-sided ideals, by showing how they all arise.

Other examples of this in Part B, concerning 'equivalence' in algebras. Part of this generalizes some of the material in Part A (joint work with Matt Neal). So the Part A and Part B will eventually be seen to be quite well connected to each other.

I. Introduction

Operator algebra $=$ closed subalgebra of $B(H)$, for a Hilbert space H

unital: has an identity of norm 1

approximately unital: has a cai

Think: 'noncomm. function algebra/uniform algebra'

or think: 'partial C^*-algebra'

- We often use C^*-algebras generated the algebra A
- or the diagonal $\Delta(A) = A \cap A^*$, a C^*-algebra
- Projection in A $=$ norm 1 (orthogonal) idempotent in A

Main theme in this talk:

generalize C^*-algebra tools and theories to operator algebras

using new ideas, such as the new 'positivity' alluded to

From a previous BA conference talk : Title: A nice class of Banach algebras

In view of the developments in the subject of operator algebras (from an operator space perspective) in the last decade or so, it seems to be a good time to reexamine the connections

$$\text{Banach algebras} \quad \leftrightarrow \quad \text{operator algebras}$$

For example, operator algebras are a 'halfway house' between (noninvolutive) Banach algebras and C^*-algebras

From a previous BA conference talk : Title: A nice class of Banach algebras

In view of the developments in the subject of operator algebras (from an operator space perspective) in the last decade or so, it seems to be a good time to reexamine the connections

$$\text{Banach algebras} \quad \leftrightarrow \quad \text{operator algebras}$$

For example, operator algebras are a 'halfway house' between (noninvolutive) Banach algebras and C^*-algebras

- It is interesting to ask questions inspired by Banach algebra theory, about operator algebras

- Conversely, it is interesting to ask about operator algebra properties which generalize important properties of C^*-algebras, which might extend to (certain) classes of Banach algebras

Many results in the present talk immediately raise the question of Banach algebra variants.

Some of them may possibly even be doable.... for classes of Banach algebras ...

(I will try point out some of these)

Every operator algebra A has a unique unitization A^1, up to completely isometric isomorphism (Ralf Meyer)

Below 1 always refers to the identity of A^1, certainly if A has no identity.

Every operator algebra A has a unique unitization A^1, up to completely isometric isomorphism (Ralf Meyer)

Below 1 always refers to the identity of A^1, certainly if A has no identity.

Many of our results have their origin in a very deep recent theorem of Read, in answer to a question of B, Hay, and Neal:

Theorem (Read) An operator algebra with a cai, has a cai (e_t) with $\|1 - e_t\| \leq 1$, and even with $\|1 - 2e_t\| \leq 1$.

(The proof is an incredible (and lengthy) noncommutative gliding hump argument)

This result drew our attention to the set $\mathfrak{S}_A$ of operators x in an operator algebra A satisfying $\|1 - 2x\| \leq 1$. Indeed the positive scalar multiples of this set form a cone $\mathfrak{c}_A$

This will play a role for us very much akin to the role of the positive cone in a C^*-algebra. This surprising claim is justified at many points in our paper, e.g. the following results of B-Read:

This result drew our attention to the set $\mathfrak{S}_A$ of operators x in an operator algebra A satisfying $\|1-2x\| \leq 1$. Indeed the positive scalar multiples of this set form a cone $\mathfrak{c}_A$

This will play a role for us very much akin to the role of the positive cone in a C^*-algebra. This surprising claim is justified at many points in our paper, e.g. the following results of B-Read:

Theorem A (not necessarily unital) linear map $T : A \to B$ between C^*-algebras or operator systems is completely positive in the usual sense iff there is a constant $C > 0$ such that $T(\mathfrak{S}_A) \subset C\mathfrak{S}_B$, and similarly at the matrix levels

This result drew our attention to the set $\mathfrak{S}_A$ of operators x in an operator algebra A satisfying $\|1 - 2x\| \leq 1$. Indeed the positive scalar multiples of this set form a cone $\mathfrak{c}_A$

This will play a role for us very much akin to the role of the positive cone in a C^*-algebra. This surprising claim is justified at many points in our paper, e.g. the following results of B-Read:

Theorem A (not necessarily unital) linear map $T : A \to B$ between C^*-algebras or operator systems is completely positive in the usual sense iff there is a constant $C > 0$ such that $T(\mathfrak{S}_A) \subset C\mathfrak{S}_B$, and similarly at the matrix levels

Definition A linear map $T : A \to B$ between operator algebras or unital operator spaces is *operator completely positive*, or OCP, if there is a constant $C > 0$ such that $T(\mathfrak{S}_A) \subset C\mathfrak{S}_B$ and similarly at the matrix levels

(Extension and Stinespring-type) Theorem A linear map $T : A \longrightarrow B(H)$ on a unital operator space or operator algebra is OCP iff T has a completely positive extension $\tilde{T} : C^*(A) \longrightarrow B(H)$

(Extension and Stinespring-type) Theorem A linear map $T : A \to B(H)$ on a unital operator space or operator algebra is OCP iff T has a completely positive extension $\tilde{T} : C^*(A) \to B(H)$

This is equivalent to being able to write T as the restriction to A of $V^*\pi(\cdot)V$ for a $*$-representation $\pi : C \to B(K)$, and an operator $V : H \to K$.

(This is just an example illustrating the principle above, our other examples will concern the algebra structure)

Moral: there is a cone $\mathfrak{c}_A$ which replaces positivity for some purposes

This cone is big (densely spans the algebra, and contains a cai)

Moral: there is a cone $\mathfrak{c}_A$ which replaces positivity for some purposes

This cone is big (densely spans the algebra, and contains a cai)

In a unital operator algebra this cone is $\mathbb{R}^+(1 + \mathrm{Ball}(A))$

So in some ways this cone is surprisingly simple (in other ways, in approximately unital algebras, it may not be so simple)

Part A. (Joint with Charles Read)

II. First consequences and lemma

Application to noncommutative peak sets: Important classical tool: for a function algebra/space A a p-set is an intersection of peak sets.

A peak set $E = f^{-1}(\{1\})$ for $f \in A, \|f\| = 1$

Equivalently, E is a peak set iff $\exists g \in A$ with $|g|_{|E^c} < \|g\| = 1 = g_{|E}$.

Part A. (Joint with Charles Read)

II. First consequences and lemma

Application to noncommutative peak sets: Important classical tool: for a function algebra/space A a p-set is an intersection of peak sets.

A peak set $E = f^{-1}(\{1\})$ for $f \in A, \|f\| = 1$

Equivalently, E is a peak set iff $\exists g \in A$ with $|g|_{|E^c} < \|g\| = 1 = g_{|E}$.

Damon Hay studied the noncommutative version of these in his PhD thesis, and got some deep results. Unfortunately there was an unpleasant dichotomy: we had two definitions of what a noncommutative p-projection should be. One is as a decreasing limit of 'peak projections' (defined later), the other is a closed projection in A^{**}, and we did not know if they coincided.

Which they do in the classical case, by a foundational theorem of Glicksberg which at that time we were unable to generalize. But we did reduce it to a question that prompted Read's theorem above. Thus the unpleasant dichotomy disappears:

Which they do in the classical case, by a foundational theorem of Glicksberg which at that time we were unable to generalize. But we did reduce it to a question that prompted Read's theorem above. Thus the unpleasant dichotomy disappears:

Theorem If A is a unital operator algebra then the p-projections are precisely the closed projections in A^{**}

This is the noncommutative version of the fundamental theorem of Glicksberg on which the theory of peak sets rests. Thus the result puts the theory of noncommutative peak sets on a much firmer foundation.

Some foundational lemmas:

Recall $\mathfrak{S}_A = \{x \in A : \|1 - 2x\| \leq 1\}$

Lemma If $x \in \mathfrak{S}_A$, with $x \neq 0$, then the operator algebra $\mathrm{oa}(x)$ generated by x has a cai

Some foundational lemmas:

Recall $\mathfrak{S}_A = \{x \in A : \|1 - 2x\| \leq 1\}$

Lemma If $x \in \mathfrak{S}_A$, with $x \neq 0$, then the operator algebra $\mathrm{oa}(x)$ generated by x has a cai

Proposition $\mathfrak{S}_A$ is closed under taking roots. That is, $x^t \in \mathfrak{S}_A$ for $0 < t \leq 1$ if $x \in \mathfrak{S}_A$. Also, $x^t \in \mathrm{oa}(x)$ (and forms a cai).

Some foundational lemmas:

Recall $\mathfrak{S}_A = \{x \in A : \|1 - 2x\| \leq 1\}$

Lemma If $x \in \mathfrak{S}_A$, with $x \neq 0$, then the operator algebra $\mathrm{oa}(x)$ generated by x has a cai

Proposition $\mathfrak{S}_A$ is closed under taking roots. That is, $x^t \in \mathfrak{S}_A$ for $0 < t \leq 1$ if $x \in \mathfrak{S}_A$. Also, $x^t \in \mathrm{oa}(x)$ (and forms a cai).

Explicitly: $x^t = \sum_{k=0}^{\infty} \binom{t}{k} (-1)^k (1 - x)^k$

Theorem An operator algebra A with cai, has, for every $\epsilon > 0$, a cai (e_t) in $\mathfrak{S}_A$, with the spectrum and numerical range of every e_t contained in the thin cigar/wedge-shaped region containing the real interval $[0, 1]$ consisting of numbers $re^{i\theta}$ with argument θ such that $|\theta| < \epsilon$, which are inside the circle $|\frac{1}{2} - z| \leq \frac{1}{2}$.

- An operator with numerical range contained in $[0, 1] \times [-\epsilon, \epsilon]$, in fact is near to a positive operator. It thus follows from the Theorem that any operator algebra with cai has a cai that gets arbitrarily close to being positive.

• An operator with numerical range contained in $[0,1] \times [-\epsilon, \epsilon]$, in fact is near to a positive operator. It thus follows from the Theorem that any operator algebra with cai has a cai that gets arbitrarily close to being positive.

A projection $p \in A^{**}$ is open if there is a net $x_t \in A$ with $x_t = px_tp \to p$ weak*.

Theorem (B-Hay-Neal) If B is a C^*-algebra containing A then $p \in A^{\perp\perp}$ is open in A^{**} iff it is open in B^{**}, i.e is open in the sense of Akemann

Lemma For any operator algebra A, if $x \in \mathfrak{S}_A$, with $x \neq 0$, then the left support projection of x equals the right support projection. In fact this projection $s(x)$ is the weak* limit of $(x^{\frac{1}{n}})$, and is an open projection.

Lemma For any operator algebra A, if $x \in \mathfrak{S}_A$, with $x \neq 0$, then the left support projection of x equals the right support projection. In fact this projection $s(x)$ is the weak* limit of $(x^{\frac{1}{n}})$, and is an open projection in the sense of Akemann.

Lemma If $x, y \in \mathfrak{S}_A$, for any operator algebra A, then $\overline{xA} \subset \overline{yA}$ iff $s(x) \leq s(y)$. If A has a cai and $x \in \mathfrak{S}_A$, then the following are equivalent:

(i) $\overline{xA} = A$.

(ii) $\overline{xAx} = A$.

(iii) $s(x) = 1_{A^{**}}$.

(iv) $\varphi(x) \neq 0$ for every state φ of A.

(v) $\mathrm{Re}(x)$ is strictly positive.

(Theorem continued) If A is unital these are equivalent to:

(vi) $\|2 - x\| < 2$

(vii) x is invertible

(viii) $1 - x$ has a nonzero peak projection

In particular,

Theorem Let T be an operator in $B(H)$ with $\|I - T\| \leq 1$. Then T is not invertible if and only if $\|I - T\| = \|I - \frac{1}{2}T\| = 1$. Also, T is invertible iff T is invertible in the closed algebra generated by I and T, and iff $\mathrm{oa}(T)$ contains I_H.

III Building cai's

We now discuss how cais are built for an operator algebra A, or for one-sided ideals or hereditary subalgebras (HSA's). Every operator algebra is an ideal/HSA of course.

An r-ideal is a right ideal with a left cai

An ℓ-ideal is a left ideal with a right cai

A HSA is an approximately unital subalgebra D of A such that $DAD \subset D$

III Building cai's

We now discuss how cais are built for an operator algebra A, or for one-sided ideals or hereditary subalgebras (HSA's). Every operator algebra is an ideal/HSA of course.

An r-ideal is a right ideal with a left cai

An ℓ-ideal is a left ideal with a right cai

A HSA is an approximately unital subalgebra D of A such that $DAD \subset D$

- There are bijective correspondence between the r-ideals, ℓ-ideals, and HSA's. But this is deep.

- The basic theory of HSA's ([B-Hay-Neal], etc) generalizes the basic C^*-algebra case, but some things are much harder.

- HSA's are in an order preserving, bijective correspondence with the open projections $p \in A^{**}$

- The basic theory of HSA's ([B-Hay-Neal], etc) generalizes the basic C^*-algebra case, but some things are much harder.

- HSA's are in an order preserving, bijective correspondence with the open projections $p \in A^{**}$

- Indeed the weak* limit of a cai for a HSA is an open projection, and is called the *support projection* of the HSA.

Conversely, if p is an open projection in A^{**}, then $\{a \in A : a = pap\}$ is a HSA in A.

- The basic theory of HSA's ([B-Hay-Neal], etc) generalizes the basic C^*-algebra case, but some things are much harder.

- HSA's are in an order preserving, bijective correspondence with the open projections $p \in A^{**}$

- Indeed the weak* limit of a cai for a HSA is an open projection, and is called the *support projection* of the HSA.

Conversely, if p is an open projection in A^{**}, then $\{a \in A : a = pap\}$ is a HSA in A.

- If $a \in \mathfrak{S}_A$ then $\overline{aAa}$ is a HSA of A, and the support projection of this HSA is $s(a)$. So $\overline{aAa} = \{x \in A : x = s(a)xs(a)\}$

Similarly, e.g. $\overline{aA} = \{x \in A : x = s(a)x)\}$. These are the 'principal' ones

Theorem Let A be any operator algebra

(1) Every separable r-ideal (resp. HSA) in A, is equal to $\overline{xA}$ (resp. $\overline{xAx}$), for some $x \in \mathfrak{S}_A$.

(2) The closure of a countable sum of r-ideals (resp. HSA's) of the form at the end of (1), is of the same form.

Theorem Let A be any operator algebra

(1) Every separable r-ideal (resp. HSA) in A, is equal to $\overline{xA}$ (resp. $\overline{xAx}$), for some $x \in \mathfrak{S}_A$.

(2) The closure of a countable sum of r-ideals (resp. HSA's) of the form at the end of (1), is of the same form.

Theorem Let A be any operator algebra (not necessarily with an identity or approximate identity). The r-ideals (resp. HSA's) in A, are precisely the closures of increasing unions of ideals (resp. HSA's) of the form $\overline{xA}$ (resp. $\overline{xAx}$), for $x \in \mathfrak{S}_A$.

Theorem Let A be any operator algebra

(1) Every separable r-ideal (resp. HSA) in A, is equal to $\overline{xA}$ (resp. $\overline{xAx}$), for some $x \in \mathfrak{S}_A$.

(2) The closure of a countable sum of r-ideals (resp. HSA's) of the form at the end of (1), is of the same form.

Theorem Let A be any operator algebra (not necessarily with an identity or approximate identity). The r-ideals (resp. HSA's) in A, are precisely the closures of increasing unions of ideals (resp. HSA's) of the form $\overline{xA}$ (resp. $\overline{xAx}$), for $x \in \mathfrak{S}_A$.

Q: Any Banach algebra variants?

(The results above rely on Read's deep theorem)

The above considerations gives an ‘algorithm’ for building approximate identities in r-ideals, ℓ-ideals, or HSA’s. In the separable case, we can just take $(x^{1/n})$ where x is as in the second last theorem, and if we want a cai simply divide by the norms

The above considerations gives an ‘algorithm’ for building approximate identities in r-ideals, ℓ-ideals, or HSA’s. In the separable case, we can just take $(x^{1/n})$ where x is as in the second last theorem, and if we want a cai simply divide by the norms

In the nonseparable case, any r-ideal J, for example, in a unital operator algebra A may written as an increasing union of r-ideals $J_t = \overline{x_t A}$. Then it is easy to see that $(x_t^{1/n})$ is a left bai for J, and again we can normalize if we want a left cai.

Corollary If A is a separable operator algebra, generating a C^*-algebra B, then the open projections in $A^{\perp\perp}$ are precisely the $s(x)$ for $x \in \mathfrak{S}_A$. In in addition A is unital, then these projections are precisely the peak projections for A

Corollary If A is a separable operator algebra, generating a C^*-algebra B, then the open projections in $A^{\perp\perp}$ are precisely the $s(x)$ for $x \in \mathfrak{S}_A$. In in addition A is unital, then these projections are precisely the peak projections for A

Corollary If A is a separable operator algebra with cai, then there exists an $x \in \mathfrak{S}_A$ with $A = \overline{xA} = \overline{Ax} = \overline{xAx}$.

Corollary If A is a separable operator algebra, generating a C^*-algebra B, then the open projections in $A^{\perp\perp}$ are precisely the $s(x)$ for $x \in \mathfrak{S}_A$. In in addition A is unital, then these projections are precisely the peak projections for A

Corollary If A is a separable operator algebra with cai, then there exists an $x \in \mathfrak{S}_A$ with $A = \overline{xA} = \overline{Ax} = \overline{xAx}$.

Corollary Any separable operator algebra with cai has a countable cai consisting of mutually commuting elements, indeed of form $(x^{\frac{1}{k}})$ for an $x \in \mathfrak{S}_A$.

These all generalize well-known C^*-algebra results, and were unobtainable previously

Another generalization of a well-known result for C^*-algebras:

Theorem Let A be any operator algebra with cai. The following are equivalent:

(i) A has a countable cai.

(ii) A has an element in $\mathfrak{S}_A$ whose real part is strictly positive.

(iii) There is an element x in $\mathfrak{S}_A$ with $s(x) = 1_{A^{**}}$.

- We get a interpolation result, generalizing Akemann's noncommutative Urysohn lemma:

• We get a interpolation result, generalizing Akemann's noncommutative Urysohn lemma:

Theorem (Nonselfadjoint noncommutative Urysohn lemma for nonselfadjoint operator algebras , 2010) Let A be a closed subalgebra of a C^*-algebra B. Given a closed projection $q \in A^{\perp\perp}$ and an open $u \geq q$ and $\epsilon > 0$, there exists $a \in A$ with $\|a\| \leq 1, aq = q$, $\|1 - 2a\| \leq 1$, and $\|a(1 - u)\| < \epsilon$.

• We had a similar result earlier [BHN], but the proof had a gap. Besides, the new one is a bit better.

IV. When xA and Ax are closed/pseudoinvertibility

IV. When xA and Ax are closed/pseudoinvertibility

The C^*-algebra result:

We recall that 'well supported' operators are those operators x that have a 'spectral gap' for $|x|$ at 0, that is 0 is absent from, or is isolated in, the spectrum of $|x|$.

IV. When xA and Ax are closed/pseudoinvertibility

The C^*-algebra result:

We recall that 'well supported' operators are those operators x that have a 'spectral gap' for $|x|$ at 0, that is 0 is absent from, or is isolated in, the spectrum of $|x|$.

Theorem An element x of a C^*-algebra A is well supported iff xA is closed, and iff there exists $y \in A$ with $xyx = x$.

IV. When xA and Ax are closed/pseudoinvertibility

The C^*-algebra result:

We recall that 'well supported' operators are those operators x that have a 'spectral gap' for $|x|$ at 0, that is 0 is absent from, or is isolated in, the spectrum of $|x|$.

Theorem (Harte-Mbekhta) An element x of a C^*-algebra A is well supported iff xA is closed, and iff there exists $y \in A$ with $xyx = x$.

Such a y is called a *generalized inverse* or *pseudoinverse*.

We get a similar result, about pseudoinvertibility in nonselfadjoint operator algebras, and with 'spectral gap' for x not $|x|$, for our cone.

Theorem For any operator algebra A, if $x \in \mathfrak{S}_A$, then the following are equivalent:

(i) $\mathrm{oa}(x)$ is unital and x is invertible in $\mathrm{oa}(x)$.

(ii) xAx is closed.

(iii) xA and Ax are closed.

(iv) There exists $y \in A$ with $xyx = x$.

Also, the latter conditions imply

(v) 0 is isolated in, or absent from, $\mathrm{Sp}_A(x)$.

The y may be taken to be in $\mathrm{oa}(x)$. Finally, if further $\mathrm{oa}(x)$ is semisimple, then conditions (i)–(v) are all equivalent.

- We do not know if xA is closed iff Ax is closed
- In a nonsemisimple setting, 0 being an isolated point in $\mathrm{Sp}(x)$ need not imply that xA is closed.

Part B. (With M. Neal)

V. Equivalence and comparison in operator algebras

To introduce the ideas in this Part (which will be eventually seen in part to be a sequel of Part A), consider a relation between two elements a and b which one may define in any monoid or algebra A: namely that there exists $x, y \in A$ with $a = xy, b = yx$

Part B. (With M. Neal)

V. Equivalence and comparison in operator algebras

To introduce the ideas in this Part (which will be eventually seen in part to be a sequel of Part A), consider a relation between two elements a and b which one may define in any monoid or algebra A: namely that there exists $x, y \in A$ with $a = xy, b = yx$

If A is a group, then this defines an equivalence relation. In an algebra this is not an equivalence relation in general.

In fact this fails to be an equivalence relation even for the case $A = M_n$, the n by n matrices. (Even in this simple case this relation is quite subtle, and is not what we want)

Part B. (With M. Neal)

V. Equivalence and comparison in operator algebras

To introduce the ideas in this Part (which will be eventually seen in part to be a sequel of Part A), consider a relation between two elements a and b which one may define in any monoid or algebra A: namely that there exists $x, y \in A$ with $a = xy, b = yx$

If A is a group, then this defines an equivalence relation. In an algebra this is not an equivalence relation in general.

In fact this fails to be an equivalence relation even for the case $A = M_n$, the n by n matrices. (Even in this simple case this relation is quite subtle, and is not what we want)

How to fix this problem in an operator algebra?

In a C^*-algebra A, the 'fix' is to insist that $y = x^*$ above; and then this defines an equivalence relation $\sim$ on the positive cone A_+ of A.

This is sometimes called *Pedersen equivalence*.

In a C^*-algebra A, the 'fix' is to insist that $y = x^*$ above; and then this defines an equivalence relation $\sim$ on the positive cone A_+ of A.

This is sometimes called *Pedersen equivalence*.

We will expand this 'fix' to a larger set than A_+, and to general operator algebras

In a C^*-algebra A, the 'fix' is to insist that $y = x^*$ above; and then this defines an equivalence relation $\sim$ on the positive cone A_+ of A.

This is sometimes called *Pedersen equivalence*.

We will expand this 'fix' to a larger set than A_+, and to general operator algebras

In C^*-algebra theory, coarser equivalence relations than Pedersen equivalence, and matching notions of subequivalence or comparison, are becoming increasingly important

For example, recently the study of Cuntz equivalence and subequivalence has become one of the most important areas of C^*-algebra theory

And *comparison theory* in C^*-algebras, where one considers coarser notions of ordering of elements in a C^*-algebra than the usual $\leq$, generalizing in some sense the crucially important comparison theory of projections in a von Neumann algebra is obviously important

Our goal in this project is to begin to transfer some portion of these tools, results, and perspectives to more general operator algebras than C^*-algebras

We use [Ortega-Rordam-Thiel] as our guide: these authors recast various equivalences and subequivalences in terms of open projections.

We try to do the same. Here we discuss generalizations of Pedersen, Blackadar, and Peligrad-Zsido equivalence .

In the sequel we hope to discuss Cuntz equivalence and subequivalence. A Cuntz semigroup is probably not going to be very useful in our setting.

Throughout A is a fixed operator algebra, with cai for simplicity, and B is a C^*-algebra containing A, and $a, b \in \mathfrak{S}_A$

We write $a \sim_c b$ in A if $a = xy, b = yx$ for contractions $x, y \in A$

Throughout A is a fixed operator algebra, with cai for simplicity, and B is a C^*-algebra containing A, and $a, b \in \mathfrak{S}_A$

We write $a \sim_c b$ in A if $a = xy, b = yx$ for contractions $x, y \in A$

This is wrong too. But a tweak of it works:

Throughout A is a fixed operator algebra, with cai for simplicity, and B is a C^*-algebra containing A, and $a, b \in \mathfrak{S}_A$

We write $a \sim_c b$ in A if $a = xy, b = yx$ for contractions $x, y \in A$

This is wrong too. But a tweak of it works:

Theorem Suppose that $a, b \in \mathfrak{S}_A$, and let $c = a^{\frac{1}{2}}$ and $d = b^{\frac{1}{2}}$. TFAE:

(i) $a \sim_c b$ as above but with $|y| = |c|$.

(ii) $a \sim_c b$ as above but with $|y| = |c|, |y^*| = |d^*|, |x| = |d|, |x^*| = |c^*|$.

(iii) $a \sim_c b$ as above but with $x = cR$ and $y = Sc$ for some contractions R, S.

(iv) For all $n \in \mathbb{N}$, there exist $x_n, y_n \in \mathrm{Ball}(A)$ with $a^{\frac{1}{n}} = x_n y_n, b^{\frac{1}{n}} = y_n x_n$, and the sequence $(y_n a)$ has a norm convergent subsequence.

(v) There exists $v \in \Delta(A^{**})$ with $s(a) = v^* v$, and $va \in A$, and $b = vav^*$.

It should be admitted that these happen much much more rarely than in the C^*-algebra case. Just like in Part A: r-ideals are much rarer in general operator algebras than in C^*-algebras–but are still interesting.

We say that a is Pedersen equivalent to b in A, and write $a \sim_A b$, if the equivalent conditions in the last theorem hold.

We say that a is Pedersen equivalent to b in A, and write $a \sim_A b$, if the equivalent conditions in the last theorem hold.

- If $a, b \in \mathrm{Ball}(A_+)$ in the case A is a C^*-algebra, then $a \sim_A b$ iff there exists $x \in A$ with $a = xx^*, b = x^*x$ (the usual Pedersen equivalence)

We say that a is Pedersen equivalent to b in A, and write $a \sim_A b$, if the equivalent conditions in the last theorem hold.

- If $a, b \in \mathrm{Ball}(A_+)$ in the case A is a C^*-algebra, then $a \sim_A b$ iff there exists $x \in A$ with $a = xx^*, b = x^*x$ (the usual Pedersen equivalence)

- In a finite dimensional operator algebra, $a \sim_A b$ iff $a^{\frac{1}{n}} \sim_c b^{\frac{1}{n}}$ for all $n \in \mathbb{N}$.

- The relation $\sim_A$ in the case that $A = M_n$ is exactly unitary equivalence of elements of $\mathfrak{S}_{M_n}$

Proposition Suppose that $r > 0$. Then $a \sim_A b$ iff $a^r \sim_A b^r$.

Corollary $\sim_A$ is an equivalence relation on $\mathfrak{S}_A$

Proposition Suppose that $r > 0$. Then $a \sim_A b$ iff $a^r \sim_A b^r$.

Corollary $\sim_A$ is an equivalence relation on $\mathfrak{S}_A$

• Is there a similar fix for a class of Banach algebras? Actually much of this work only uses existence of roots with reasonable properties... and there is a theory of roots in Banach algebras (that I don't know much about)

• Of course Banach algebraists usually would be interested in quite different 'equivalences'. The key point is that we are looking for a fix that generalizes the C^*-algebra case.

Proposition Suppose that $r > 0$. Then $a \sim_A b$ iff $a^r \sim_A b^r$.

Corollary $\sim_A$ is an equivalence relation on $\mathfrak{S}_A$

- Is there a similar fix for a class of Banach algebras? Actually much of this work only uses existence of roots with reasonable properties... and there is a theory of roots in Banach algebras (that I don't know much about)

- Of course Banach algebraists usually would be interested in quite different 'equivalences'. The key point is that we are looking for a fix that generalizes the C^*-algebra case.

- Before we turn to coarser equivalence relations, we turn to an algebraic topic... which will start connecting us back to Part A...

Theorem (again) Suppose that $a, b \in \mathfrak{S}_A$, and let $c = a^{\frac{1}{2}}$ and $d = b^{\frac{1}{2}}$. TFAE:

(i) $a \sim_c b$ as above but with $|y| = |c|$.

(ii) $a \sim_c b$ as above but with $|y| = |c|, |y^*| = |d^*|, |x| = |d|, |x^*| = |c^*|$.

(iii) $a \sim_c b$ as above but with $x = cR$ and $y = Sc$ for some contractions R, S.

(iv) For all $n \in \mathbb{N}$, there exist $x_n, y_n \in \mathrm{Ball}(A)$ with $a^{\frac{1}{n}} = x_n y_n, b^{\frac{1}{n}} = y_n x_n$, and the sequence $(y_n a)$ has a norm convergent subsequence.

(v) There exists $v \in \Delta(A^{**})$ with $s(a) = v^*v, s(b) = vv^*$, and $va \in A$, and $b = vav^*$.

The v in (v), will turn out to be what we call a $*$-open partial isometry, and these are crucial in the study of $\sim_A$. And they may be characterized purely algebraically in terms of what we call a hereditary bimodule.

A hereditary bimodule is what you get, if you carefully examine the algebraic structure of aAb, when $a \sim_A b$. E.g. $\overline{aAa} \cong \overline{bAb}$, and $\overline{aAb} \cong \overline{aAa}$, and $\overline{aAb} \cong \overline{bAa}$ completely isometrically. We call the last isomorphism $\sharp$.

VI. Hereditary bimodules and open partial isometries

- We define a hereditary bimodule to be the 1-2 corner X of a HSA in $M_2(A)$, where that HSA is the linking algebra for a strong Morita equivalence (in sense of B-Muhly-Paulsen), such that X is completely isometric to an approximately unital operator algebra C. We call X a hereditary D-E-bimodule if the diagonal corners of the HSA are D and E

VI. Hereditary bimodules and open partial isometries

- We define a hereditary bimodule to be the 1-2 corner X of a HSA in $M_2(A)$, where that HSA is the linking algebra for a strong Morita equivalence (in sense of B-Muhly-Paulsen), such that X is completely isometric to an approximately unital operator algebra C. We call X a hereditary D-E-bimodule if the diagonal corners of the HSA are D and E

VI. Hereditary bimodules and open partial isometries

• We define a hereditary bimodule to be the 1-2 corner X of a HSA in $M_2(A)$, where that HSA is the linking algebra for a strong Morita equivalence (in sense of B-Muhly-Paulsen), such that X is completely isometric to an approximately unital operator algebra C. We call X a hereditary D-E-bimodule if the diagonal corners of the HSA are D and E

Notation: $A_a = \overline{aAa}$

• We define a principal hereditary bimodule to be a hereditary A_a-A_b-bimodule X for some $a, b \in \mathfrak{S}_A$

- An abstract characterization of hereditary bimodules:

Theorem A subspace X in A is a hereditary bimodule iff $XAX \subset X$, and there are nets of contractions $c_n \in X$ and $d_n \in A$, such that

(1) $c_n d_n x \to x$ and $x d_n c_n \to x$ in norm for all $x \in X$,

(2) $(d_n c_m)$ and $(c_m d_n)$ are norm convergent with n for fixed m, and are norm convergent with m for fixed n.

Proposition $a \sim_A b$ iff there exists a principal hereditary A_a-A_b-bimodule, such that $a \oplus 0$ and $0 \oplus b$ in the linking algebra of the bimodule correspond to $c \oplus 0$ and $0 \oplus c$ for some element $c \in C$. Here C is the algebra isometric to the bimodule.

Proposition $a \sim_A b$ iff there exists a principal hereditary A_a-A_b-bimodule, such that $a \oplus 0$ and $0 \oplus b$ in the linking algebra of the bimodule correspond to $c \oplus 0$ and $0 \oplus c$ for some element $c \in C$. Here C is the algebra isometric to the bimodule.

- A tripotent is a partial isometry (in B^{**} in this case)

Proposition $a \sim_A b$ iff there exists a principal hereditary A_a-A_b-bimodule, such that $a \oplus 0$ and $0 \oplus b$ in the linking algebra of the bimodule correspond to $c \oplus 0$ and $0 \oplus c$ for some element $c \in C$. Here C is the algebra isometric to the bimodule.

- A tripotent is a partial isometry (in B^{**} in this case)

- In [B-Neal, Studia Math 2007] we defined and studied *open tripotents*. There are many equivalent definitions.

- For example, u is an open tripotent iff

$$\hat{u} = \frac{1}{2} \begin{bmatrix} uu^* & u \\ u^* & u^*u \end{bmatrix}$$

is an open projection in the earlier sense (of Akemann)

- Definition. A $*$-open tripotent for an operator algebra A, is a tripotent u such that $\hat{u}$ is an open projection in $M_2(A^{**})$ in the sense of [B-Hay-Neal] described in Part A.

• Definition. A $*$-open tripotent for an operator algebra A, is a tripotent u such that $\hat{u}$ is an open projection in $M_2(A^{**})$ in the sense of [B-Hay-Neal] described in Part A.

(There is a more general notion in the second dual of a Morita equvalence bimodule, but it works out just the same, because we can just view it in the linking algebra.)

• Definition. A $*$-open tripotent for an operator algebra A, is a tripotent u such that $\hat{u}$ is an open projection in $M_2(A^{**})$ in the sense of [B-Hay-Neal] described in Part A.

(There is a more general notion in the second dual of a Morita equvalence bimodule, but it works out just the same, because we can just view it in the linking algebra.)

• The theory of these generalizes the theory of open tripotents from [B-Neal, Studia Math 2007]

• For a $*$- open tripotent v the space $A_v = \{a \in A : a = vv^*av^*v\}$ plays the role of the HSA determined by an open projection in Part A.

• One may also characterize $*$- open tripotents in terms of A_v, i.e. iff v is a weak* limit of elements in A_v, but one also has to throw in a couple of conditions like: $A_v v^* \subset A$

Proposition If v is a $*$-open tripotent in A^{**}, then A_v is a hereditary bimodule.

Conversely, any hereditary bimodule equals A_v for a $*$-open tripotent v.

• So $*$-open tripotents are essentially the same thing as hereditary bimodules. Just like open projections are essentially the same thing as HSA's (or as right ideals with a left cai)

- If X is a hereditary bimodule, so $X = A_v$ for a $*$-open tripotent, then the 'dual module' Y in the 1-2 corner of the linking algebra), is simply v^*Xv^*. Thus we can define a complete isometry from X onto Y by $x \mapsto x^\sharp = v^*xv^*$. In this language:

• If X is a hereditary bimodule, so $X = A_v$ for a $*$-open tripotent, then the 'dual module' Y in the 1-2 corner of the linking algebra), is simply $v^* X v^*$. Thus we can define a complete isometry from X onto Y by $x \mapsto x^\sharp = v^* x v^*$. In this language:

Corollary $a \sim_A b$ if and only if $\overline{aAb}$ is a hereditary bimodule, and there exists $x \in \overline{aAb}$ such that $a = xx^\sharp, b = x^\sharp x$.

• If X is a hereditary bimodule, so $X = A_v$ for a $*$-open tripotent, then the 'dual module' Y in the 1-2 corner of the linking algebra), is simply $v^* X v^*$. Thus we can define a complete isometry from X onto Y by $x \mapsto x^\sharp = v^* x v^*$. In this language:

Corollary $a \sim_A b$ if and only if $\overline{aAb}$ is a hereditary bimodule, and there exists $x \in \overline{aAb}$ such that $a = xx^\sharp, b = x^\sharp x$.

Now we turn to coarser equivalence relations...

• We define $a \cong b$ if $s(a) = s(b)$, or equivalently if $\overline{aA} = \overline{bA}$, or equivalently if $\overline{aAa} = \overline{bAb}$.

Blackadar type equivalence: We define $a \sim_s b$ if there exist $a', b' \in \mathfrak{S}_A$, with $a \cong a', a' \sim_A b'$, and $b' \cong b$.

• It will follow from a result below that this is an equivalence relation.

- We will define, for open projections $p, q \in A^{**}$, $p \sim_{A,PZ} q$ if there is a $*$-open tripotent u with $u^*u = p, uu^* = q$.

In the C^*-algebra case this turns out to coincide with an equivalence of open projections introduced by Peligrad and Zsido.

- We will define, for open projections $p, q \in A^{**}$, $p \sim_{A,PZ} q$ if there is a $*$-open tripotent u with $u^*u = p, uu^* = q$.

In the C^*-algebra case this turns out to coincide with an equivalence of open projections introduced by Peligrad and Zsido.

- Parts of the following generalizes results of Lin and Ortega-Rordam-Lin:

Theorem: TFAE:

(i) $a \sim_s b$.

(ii) $\overline{aA} \cong \overline{bA}$, completely isometrically via an A-module map.

(iii) $s(a) \sim_{A,PZ} s(b)$.

(iv) There exists $b' \in \mathfrak{c}_A$, with $a \sim_A b'$ and $b' \cong b$.

(v) There exists $a' \in \mathfrak{c}_A$, with $a \cong a' \sim_A b'$.

(vi) There exists a principal hereditary A_a-A_b-bimodule.

VII. Open tripotents and the structure of hereditary bimodules

- If A_v is a subset of A_w (resp. equals A_w), it does not follow that $v \leq w$ (resp. $v = w$). Here are some ways to fix this:

VII. Open tripotents and the structure of hereditary bimodules

• If A_v is a subset of A_w (resp. equals A_w), it does not follow that $v \leq w$ (resp. $v = w$). Here are some ways to fix this:

Lemma If v, w are $*$-open tripotents in A^{**}, then TFAE:

(i) $v \leq w$ in the usual ordering of tripotents ($vv^*w = v$),

(ii) A_v is a HSA of A_w,

(iii) $\mathfrak{S}_{A_v} \subset \mathfrak{S}_{A_w}$ as sets (where e.g. $\mathfrak{S}_{A_v} = \{a \in A_v : \|v - 2a\| \leq 1\}$),

(iv) $\hat{v} \leq \hat{w}$.

VII. Open tripotents and the structure of hereditary bimodules

- If A_v is a subset of A_w (resp. equals A_w), it does not follow that $v \leq w$ (resp. $v = w$). Here are some ways to fix this:

Lemma If v, w are $*$-open tripotents in A^{**}, then TFAE:

(i) $v \leq w$ in the usual ordering of tripotents $(vv^*w = v)$,

(ii) A_v is a HSA of A_w,

(iii) $\mathfrak{S}_{A_v} \subset \mathfrak{S}_{A_w}$ as sets (where e.g. $\mathfrak{S}_{A_v} = \{a \in A_v : \|v - 2a\| \leq 1\}$),

(iv) $\hat{v} \leq \hat{w}$.

Lemma A family $\{u_i\}$ of $*$-open tripotents in A^{**}, which are all dominated by a fixed tripotent, has a least upper bound, and this is a $*$-open tripotent in A^{**}.

Lemma Let v be a $*$-open tripotent in A^{**} and $x \in \mathfrak{S}_{A_v}$. The hereditary bimodule corresponding to the tripotent $s = s_v(x)$ is a principal hereditary bimodule, and may be written as $\overline{aAb}$ for some $a, b \in \mathfrak{S}_A$ with $a \sim_A b$

Lemma Let v be a $*$-open tripotent in A^{**} and $x \in \mathfrak{S}_{A_v}$. The hereditary bimodule corresponding to the tripotent $s = s_v(x)$ is a principal hereditary bimodule, and may be written as $\overline{aAb}$ for some $a, b \in \mathfrak{S}_A$ with $a \sim_A b$

Corollary Every hereditary bimodule in A is the closure of an increasing net of principal hereditary bimodules of the form $\overline{aAb}$, where $a, b \in \mathfrak{S}_A$ with $a \sim_A b$.

Corollary Any separable hereditary bimodule in A is a principal hereditary bimodule, that is, of the form $\overline{aAb}$, where $a, b \in \mathfrak{S}_A$ with $a \sim_A b$.

Lemma Let v be a $*$-open tripotent in A^{**} and $x \in \mathfrak{S}_{A_v}$. The hereditary bimodule corresponding to the tripotent $s = s_v(x)$ is a principal hereditary bimodule, and may be written as $\overline{aAb}$ for some $a, b \in \mathfrak{S}_A$ with $a \sim_A b$

Corollary Every hereditary bimodule in A is the closure of an increasing net of principal hereditary bimodules of the form $\overline{aAb}$, where $a, b \in \mathfrak{S}_A$ with $a \sim_A b$.

Corollary Any separable hereditary bimodule in A is a principal hereditary bimodule, that is, of the form $\overline{aAb}$, where $a, b \in \mathfrak{S}_A$ with $a \sim_A b$.

- These use the matching result for HSA's from Part A.

Subequivalence: We define $a \precsim_s b$ if there exists $b' \in \overline{bAb}$ such that $a \sim_s b'$. This is clearly equivalent to: there exists $b' \in \overline{bAb}$ such that $a \sim_A b'$. We call this *Blackadar comparison in* A.

• If p, q are open projections in A^{**} we say that $p \precsim_{A,PZ} q$ if there is an open projection $q' \leq q$ in A^{**} with $p \sim_{A,PZ} q'$. We call this *Peligrad-Zsido subequivalence in* A^{**}.

The following is the version of a result in [Ortega-Rordam-Thiel] in our setting:

Corollary If $a, b \in \mathfrak{c}_A$, TFAE:

(i) $a \precsim_s b$.

(ii) $p_a \precsim_{A,PZ} p_b$.

(iii) There exist a pair of completely isometric A-module maps $\Phi : \overline{aA} \to \overline{bA}$ and $\Psi : \overline{Aa} \to \overline{Ab}$, such that $\Psi(x)\Phi(y) = xy$ for all $x \in \overline{Aa}, y \in \overline{aA}$.