Hyperreflexivity of the derivation space of some group algebras

J. Alaminos
(joint work with J. Extremera and A. R. Villena)

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Late 2009...

Theorem (J. Alaminos, J. Extremera, and A. R. Villena)

The derivation space $\mathrm{Der}(L^1(G))$ *is a hyperreflexive subspace of* $\mathcal{B}(L^1(G))$ *in the following case:*

- G *is an* amenable *SIN-group.*

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- *G is an* amenable *SIN-group.*

- There are SIN-groups which are not amenable.
- There are important classes of groups which are both SIN and amenable, namely:
  1. *Abelian groups*
  2. *Compact groups*
  3. *Moore groups*
  4. $[FIA]^-$

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Problem (E. Kaniuth)

Is the derivation space $\mathrm{Der}(L^1(G))$ hyperreflexive for each discrete group G?

- Th
- Th ... able, namely:
  1. *Abelian groups*
  2. *Compact groups*
  3. *Moore groups*
  4. *[FIA]*$^-$

Outline

Section 1 | Introduction to hyperreflexivity

Reflexivity

D. Sarason (1966)

A subalgebra $\mathcal{A} \subset \mathcal{B}(H)$, for a complex Hilbert space H, is *reflexive* if

$$\mathcal{A} \; = \; \mathrm{alg}\Big(\mathrm{lat}(\mathcal{A})\Big)$$

i.e., the set of operators leaving all $\mathcal{A}$-invariant subspaces invariant.

Reflexivity

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A. Loginov and V. Shulman (1975)

Let X and Y be Banach spaces and let $\mathcal{M}$ be a linear subspace of $\mathcal{B}(X, Y)$. We define the *reflexive hull* of $\mathcal{M}$ by

$$\operatorname{ref}\big(\mathcal{M}\big) = \Big\{ T \in \mathcal{B}(X, Y) \colon \; Tx \in \overline{\{Sx \colon S \in \mathcal{M}\}} \; \forall x \in X \Big\}.$$

The space $\mathcal{M}$ is *reflexive* if $\operatorname{ref}\big(\mathcal{M}\big) = \mathcal{M}$.

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The space $\mathcal{M}$ is *reflexive* if $\mathrm{ref}(\mathcal{M}) = \mathcal{M}$.

If $\mathcal{A}$ is a unital algebra of operators, then $\mathrm{ref}(\mathcal{A}) \;=\; \mathrm{alg}\Big(\mathrm{lat}(\mathcal{A})\Big)$.

Reflexivity

Usual choices for $\mathcal{M}$

1. $\mathcal{M} = \mathrm{Iso}(X, Y) = \left\{\text{isometries from } X \text{ into } Y\right\}.$

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Reflexivity

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2. $\mathcal{M} = \text{Hom}(A, B) = \Big\{\text{homomorphisms from } A \text{ to } B\Big\}$, where A and B are Banach algebras.
3. $\mathcal{M} = \text{Der}(A) = \Big\{\text{derivations on } A\Big\}$, where A is a Banach algebra.

D. R. Larson (1988)

Which Banach algebras A have a reflexive derivation space $\text{Der}(A)$?

Reflexivity

of the derivation space in Operator algebras

1990 Kadison
1990 Larson, Sourour
1994 Shulman
1996 Crist
2000 Johnson
2001 Schweizer
2004 Hadwin, Li
2005 Samei

Local derivations on von Neumann algebras

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Reflexivity

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Local derivations and local automorphisms on $\mathcal{B}(X)$

Reflexivity

of the derivation space in Operator algebras

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Theorem

The space $\mathrm{Der}(A)$ *is a reflexive subspace of* $\mathcal{B}(A)$ *for each* C^**-algebra* A*.*

Reflexivity

Group algebras

2005 Samei

2007 Alaminos, Brešar, Extremera, and Villena

2010 Alaminos, Extremera, and Villena

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$\mathrm{Der}(L^1(G))$ is reflexive when G is a SIN or a totally disconnected group

Reflexivity

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2005 Samei

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Some positive results for C^*-algebras

Reflexivity

Group algebras

2005 Samei

2007 Alaminos, Brešar, Extremera, and Villena

2010 Alaminos, Extremera, and Villena

2011 Samei

Reflexivity of the space derivations of a group algebra on an amenable SIN group

Reflexivity

Group algebras

2005 Samei

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Reflexivity of $\mathrm{Der}(L^1(G))$ when G has an open subgroup with polynomial growth

Reflexivity

Group algebras

Theorem (E. Samei, 2011)

The space $\mathrm{Der}(L^1(G))$ *is a reflexive subspace of* $\mathcal{B}(L^1(G))$ *in any of the following cases:*

- *G is a PG-group*
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Problem

For which locally compact groups G is the space $\mathrm{Der}\big(L^1(G)\big)$ reflexive?

Hyperreflexivity

Arveson (1975), Kraus, Larson (1986)

Let X and Y be Banach spaces and let $\mathcal{M}$ be a linear subspace of $\mathcal{B}(X, Y)$. For every $T \in \mathcal{B}(X, Y)$, we define

$$\operatorname{dist}_r(T, \mathcal{M}) = \sup_{\|x\| \leq 1} \inf_{S \in \mathcal{M}} \|Tx - Sx\|.$$

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The space $\mathcal{M}$ is *reflexive* if $\operatorname{dist}_r(T, \mathcal{M}) = 0 \Rightarrow \operatorname{dist}(T, \mathcal{M}) = 0$.

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The space $\mathcal{M}$ is *hyperreflexive* if there exists a constant $C > 0$ such that

$$\operatorname{dist}(T, \mathcal{M}) \leq C \operatorname{dist}_r(T, \mathcal{M})$$

for every $T \in \mathcal{B}(X, Y)$.

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Hyperreflexivity

of the derivation space

Problem

- Which Banach algebras A have a hyperreflexive derivation space $\mathrm{Der}(A)$?

Hyperreflexivity

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Problem

- Which Banach algebras A have a hyperreflexive derivation space $\mathrm{Der}(A)$?
- For which C^*-algebras A is $\mathrm{Der}\big(A\big)$ hyperreflexive?

Theorem (V. Shulman, 1994)

The space $\mathrm{Der}(A)$ *is a hyperreflexive subspace of* $\mathcal{B}(A)$ *for each* C^**-algebra* A *with* $H^2(A, A) = \{0\}$.

Hyperreflexivity

Group algebras

Group algebras

2010 J. Alaminos, J. Extremera, and A. R. Villena

2011 E. Samei

Hyperreflexivity

Group algebras

Group algebras

2010 J. Alaminos, J. Extremera, and A. R. Villena

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Problem

For which locally compact groups G is $\mathrm{Der}\big(L^1(G)\big)$ hyperreflexive?

Hyperreflexivity

Group algebras

Theorem (E. Samei, 2011)

The space $\mathrm{Der}(L^1(G))$ *is a hyperreflexive subspace of* $\mathcal{B}(L^1(G))$ *in any of the following cases:*

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Hyperreflexivity

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Section 2 | Approximate derivations

Approximate derivations

Derivations

$$D(ab) - D(a)b - aD(b) \;=\; 0 \quad (a, b \in A)$$

Approximate derivations

Derivations

$$D(ab) - D(a)b - aD(b) \;=\; 0 \qquad (a, b \in A)$$

Measuring derivativity of $T \in \mathcal{B}(A)$

$$\operatorname{der}(T) = \sup\left\{\left\| T(ab) - T(a)b - aT(b)\right\| : \; a, b \in S_A\right\}$$

Approximate derivations

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$$\operatorname{der}(T) \leq 3 \operatorname{dist}\Big(T, \operatorname{Der}(A)\Big)$$

Approximate derivations

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Question

$$\operatorname{der}(T) \text{ being small } \implies \operatorname{dist}(T, \operatorname{Der}(A)) \text{ being small?}$$

Approximate derivations

Our answer

Theorem

Let G be an arbitrary locally compact group. Then there exists a constant $C > 0$ with the property that

$$\operatorname{dist}\left(T, \operatorname{Der}(L^1(G))\right) \leq C \operatorname{der}(T) \quad (T \in \mathcal{B}(L^1(G))).$$

Approximate derivations

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Tools

1. Ultraproducts

Approximate derivations

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Tools

1. Ultraproducts
2. A fixed point theorem for L^1 spaces

Sketch of the proof

Assume towards a contradiction that the theorem is false.

- There exists a bounded sequence (T_n) in $\mathcal{B}(L^1(G))$ with

$$\lim_{n\to\infty} \operatorname{der}(T_n) = 0 \text{ and } \operatorname{dist}\big(T_n, \operatorname{Der}\big(L^1(G)\big)\big) = 1.$$

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- We then give rise to a bounded sequence $(\overline{T}_n)$ in $\mathcal{B}\big(M(G)\big)$ such that

$$\lim_{n\to\infty} \operatorname{der}(\overline{T}_n) = 0 \text{ and } \lim_{n\to\infty} \left\| \overline{T}_{n|L^1(G)} - T_n \right\| = 0.$$

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- Let us consider

$$\mathbf{D} = (\overline{T}_n)^{\mathcal{U}} : \big(M(G)\big)^{\mathcal{U}} \to \big(M(G)\big)^{\mathcal{U}}, \quad \mathbf{D}(\mathbf{u}) = \big(\overline{T}_n(\mu_n)\big)^{\mathcal{U}} \quad .$$

for $\mathbf{u} = (\mu_n)^{\mathcal{U}} \in (M(G))^{\mathcal{U}}$.
Then $\mathbf{D}$ is a derivation on the Banach algebra $\big(M(G)\big)^{\mathcal{U}}$.

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- We then give rise to a bounded sequence $(\overline{T}_n)$ in $\mathcal{B}(M(G))$ such that

Theorem (U. Bader, T. Gelander, and N. Monod, 2010)

Let K be a non-empty bounded subset of an L-embedded Banach space X. Let $\mathcal{A}$ be a set of affine isometries of X preserving K. Then there is a point $u \in X$ such that $T(u) = u$ for each $T \in \mathcal{A}$.

$$\mathbf{D} = (\overline{T}_n)^{\mathcal{U}} : \left(M(G)\right)^{\mathcal{U}} \to \left(M(G)\right)^{\mathcal{U}}, \quad \mathbf{D}(\mathbf{u}) = \left(\overline{T}_n(\mu_n)\right)^{\mathcal{U}} .$$

for $\mathbf{u} = (\mu_n)^{\mathcal{U}} \in (M(G))^{\mathcal{U}}$.

Then $\mathbf{D}$ is a derivation on the Banach algebra $\left(M(G)\right)^{\mathcal{U}}$.

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- We then give rise to a bounded sequence $(\overline{T}_n)$ in $\mathcal{B}\big(M(G)\big)$ such that

$$\lim_{n\to\infty} \operatorname{der}(\overline{T}_n) = 0 \text{ and } \lim_{n\to\infty} \left\|\overline{T}_{n|L^1(G)} - T_n\right\| = 0.$$

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- There exists a point $\mathbf{u} = (\mu_n) \in \big(M(G)\big)^{\mathcal{U}}$, which satisfies

$$\mathbf{D}(\delta_{\mathbf{t}}) = \mathbf{u}\delta_{\mathbf{t}} - \delta_{\mathbf{t}}\mathbf{u} \quad (\mathbf{t} \in \mathbf{G}) \implies \lim_{\mathcal{U}} \left\| T_n - \operatorname{ad}(\mu_n)\right\| = 0 \;\lightning$$

Section 3 | Approximate generalized derivations

Approximate generalized derivations

Generalized derivations

$$T(abc) - T(ab)c - aT(bc) + aT(b)c \;=\; 0 \;\; (a, b, c \in A)$$

Approximate generalized derivations

Generalized derivations

$$T(abc) - T(ab)c - aT(bc) + aT(b)c \ = \ 0 \ \ (a, b, c \in A)$$

Generalized derivations on group algebras

$$\mathrm{GDer}(L^1(G)) = \mathrm{Der}(L^1(G)) + \mathrm{M}_\ell(L^1(G))$$

Approximate generalized derivations

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Generalized derivations on group algebras

$$\mathrm{GDer}(L^1(G)) = \mathrm{Der}(L^1(G)) + \mathrm{M}_\ell(L^1(G))$$

Measuring the generalized derivativity of $T \in \mathcal{B}(A)$

$$\mathrm{gder}(T) \;=\; \sup\left\{\left\|T(abc) - T(ab)c - aT(bc) + aT(b)c\right\| : \; a, b, c \in S_A\right\}$$

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Theorem

$$\mathrm{dist}\Big(T, \mathrm{GDer}(L^1(G))\Big) \leq C\, \mathrm{gder}(T) \qquad (T \in \mathcal{B}(L^1(G))).$$

Section 4 | Putting all together

First attempt

Theorem

Let G be an arbitrary locally compact group. Then $\mathrm{GDer}(L^1(G))$ is hyperreflexive.

Proof

Ingredients

Tools

Proof

Ingredients

Tools

1. “Ultra”-techniques

Proof

Ingredients

Tools

1. "Ultra"-techniques
2. Synthesis

Proof

Ingredients

Tools

1. "Ultra"-techniques
2. Synthesis
3. Approximate generalized derivations

Proof

Sketch

Assume towards a contradiction that the theorem is false.

- There exists a bounded sequence (T_n) in $\mathcal{B}\big(L^1(G)\big)$ with

$$\lim_{n\to\infty} \operatorname{dist}_r\Big(T_n, \operatorname{GDer}(L^1(G))\Big) = 0 \quad \text{and} \quad \operatorname{dist}\Big(T_n, \operatorname{GDer}(L^1(G))\Big) = 1.$$

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- Pick sequences (s_n) and (t_n) in G, a bounded sequence (g_n) in $L^1(G)$, and a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define $\Phi\colon A(\mathbb{T}^4) \to \big(L^1(G)\big)^{\mathcal{U}}$, by

$$\Phi(F) = \left(\sum_{k_1,k_2,k_3,k_4\in\mathbb{Z}} \widehat{F}(k_1,k_2,k_3,k_4)\ \delta_{s_n}^{k_1} * T_n\big(\delta_{s_n}^{k_2} * g_n * \delta_{t_n}^{k_3}\big) * \delta_{t_n}^{k_4} \right)^{\mathcal{U}}$$

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- Φ satisfies the property

$$F \in A\big(\mathbb{T}^4\big),\ \operatorname{supp}(F) \cap \underbrace{\big\{z \in \mathbb{T}^4 \colon\ z_1 = z_2 \text{ or } z_3 = z_4\big\}}_{\Delta} = \emptyset \implies \Phi(F) = 0$$

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$$\Phi(F) = \left(\sum_{k_1,k_2,k_3,k_4\in\mathbb{Z}} \widehat{F}(k_1,k_2,k_3,k_4)\ \delta_{s_n}^{k_1} * T_n\big(\delta_{s_n}^{k_2} * g_n * \delta_{t_n}^{k_3}\big) * \delta_{t_n}^{k_4} \right)^{\mathcal{U}}$$

- Φ satisfies the property

$$\lim_{\mathcal{U}} \big\| \delta_{s_n} * T_n\left(g_n * \delta_{t_n}\right) - T_n\left(\delta_{s_n} * g_n * \delta_{t_n}\right) - \delta_{s_n} * T_n\left(g_n\right) * \delta_{t_n} + T_n\left(\delta_{s_n} * g_n\right) * \delta_{t_n} \big\| = 0$$

Proof

Sketch

Assume towards a contradiction that the theorem is false.

- There exists a bounded sequence (T_n) in $\mathcal{B}(L^1(G))$ with

$$\lim_{n\to\infty} \operatorname{dist}_r\Big(T_n, \operatorname{GDer}(L^1(G))\Big) = 0 \quad \text{and} \quad \operatorname{dist}\Big(T_n, \operatorname{GDer}(L^1(G))\Big) = 1.$$

- Φ satisfies the property

$$\lim_{\mathcal{U}} \left\| \delta_{s_n} * T_n(g_n * \delta_{t_n}) - T_n(\delta_{s_n} * g_n * \delta_{t_n}) - \delta_{s_n} * T_n(g_n) * \delta_{t_n} + T_n(\delta_{s_n} * g_n) * \delta_{t_n} \right\| = 0$$

- $\lim_{\mathcal{U}} \operatorname{gder}(T_n) = 0$, a contradiction ↯

First attempt

Why $\mathrm{GDer}(L^1(G))$ instead of $\mathrm{Der}(L^1(G))$?

Characterizing maps through the zero products

- *Homomorphisms:* $ab = 0 \Rightarrow \Phi(a)\Phi(b) = 0$.

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- *Derivations:* $ab = bc = 0 \Rightarrow aD(b)c = 0.$

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Characterizing maps through the zero products

- *Homomorphisms:* $ab = 0 \Rightarrow \Phi(a)\Phi(b) = 0$.
- *Derivations:* $ab = bc = 0 \Rightarrow aD(b)c = 0$.

We are not able to distinguish

$$\mathrm{Der}(L^1(G))$$

from

$$\mathrm{GDer}(L^1(G))$$

through a zero product analysis.

First attempt

Hyperreflexivity after the first attempt

$$\operatorname{dist}\Big(T, \operatorname{Der}(L^1(G)) + \mathsf{M}_\ell(L^1(G))\Big) \le \; C \operatorname{dist}_r\Big(T, \operatorname{Der}(L^1(G)) + \mathsf{M}_\ell(L^1(G))\Big)$$

First attempt

Hyperreflexivity after the first attempt

$$\begin{aligned}\operatorname{dist}\Big(T, \operatorname{Der}(L^1(G)) + \mathrm{M}_\ell(L^1(G))\Big) &\leq \; C \operatorname{dist}_r\Big(T, \operatorname{Der}(L^1(G)) + \mathrm{M}_\ell(L^1(G))\Big) \\ &\leq C \operatorname{dist}_r\Big(T, \operatorname{Der}(L^1(G))\Big)\end{aligned}$$

First attempt

Hyperreflexivity after the first attempt

$$\operatorname{dist}\Big(T, \operatorname{Der}(L^1(G)) + \mathrm{M}_\ell(L^1(G))\Big) \leq \ C \operatorname{dist}_r\Big(T, \operatorname{Der}(L^1(G)) + \mathrm{M}_\ell(L^1(G))\Big)$$
$$\leq C \operatorname{dist}_r\Big(T, \operatorname{Der}(L^1(G))\Big)$$

How do we remove the remaining $\mathrm{M}_\ell(L^1(G))$?

Second attempt

If $L^1(G)$ has an identity...

With an identity $\mathbf{1}$ in $L^1(G)$,

$$\begin{aligned}\operatorname{dist}\Big(T, \operatorname{Der}(L^1(G))\Big) &\leq 2\operatorname{dist}\Big(T, \operatorname{GDer}(L^1(G))\Big) + \|T(\mathbf{1})\| \\ &\leq 2\operatorname{dist}\Big(T, \operatorname{GDer}(L^1(G))\Big) + \operatorname{dist}_r\Big(T, \operatorname{Der}(L^1(G))\Big) \\ &\leq (2C+1)\operatorname{dist}_r\Big(T, \operatorname{Der}(L^1(G))\Big)\end{aligned}$$

Second attempt

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dist

$$\leq (2C+1)\,\mathrm{dist}_r\Big(T, \mathrm{Der}(L^1(G))\Big)$$

The answer to Kaniuth question

The derivation space $Der(L^1(G))$ is hyperreflexive for each discrete group G

Third attempt

G has an open subgroup which has polynomial growth

Lemma (E. Samei, 2011)

Let G be a locally compact group with an open subgroup which has polynomial growth. Then there are approximate identities $(\varrho_i)_{i\in I}$ and $(\xi_i)_{i\in I}$ in $L^1(G)$ bounded by $M > 0$ such that

$$\varrho_i * \xi_i = \xi_i * \varrho_i = \varrho_i \quad (i \in I).$$

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$$\Downarrow$$

$$\operatorname{dist}\Big(T, \operatorname{Der}(L^1(G))\Big) \leq \big(C(M+1)^2 + M^2 + M\big) \operatorname{dist}_r\Big(T, \operatorname{Der}(L^1(G))\Big)$$

Third attempt

G has an open subgroup which has polynomial growth

Lemma (E. Samei, 2011)

Let G be a locally compact group with an open subgroup which has polynomial growth. ...)

bounde

This works for

- PG-groups;
- IN-groups;
- MAP-groups;
- Totally disconnected groups.

$$\operatorname{dist}\Big(T, \operatorname{Der}(L^1(G))\Big) \leq \left(C(M+1)^2 + M^2 + M\right) \operatorname{dist}_r\Big(T, \operatorname{Der}(L^1(G))\Big)$$

Fourth attempt - Work in progress

With unital primitive quotients

- Since derivations on $L^1(G)$ leave the primitive ideals invariant, it follows that T also does and therefore that T drops to the quotient.
- If the quotient is unital, then we can argue as in the first case.
- This works for the SSS-groups (strongly semisimple groups): the intersection of all maximal modular ideals of $L^1(G)$ is zero.

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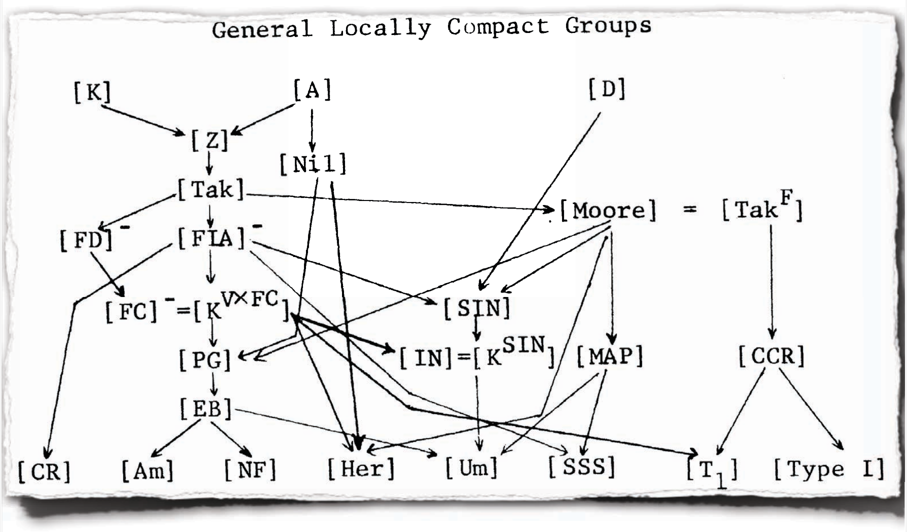
- If the quotient is unital, then we can argue as in the first case.
- This works for the SSS-groups (strongly semisimple groups): the intersection of all maximal modular ideals of $L^1(G)$ is zero.

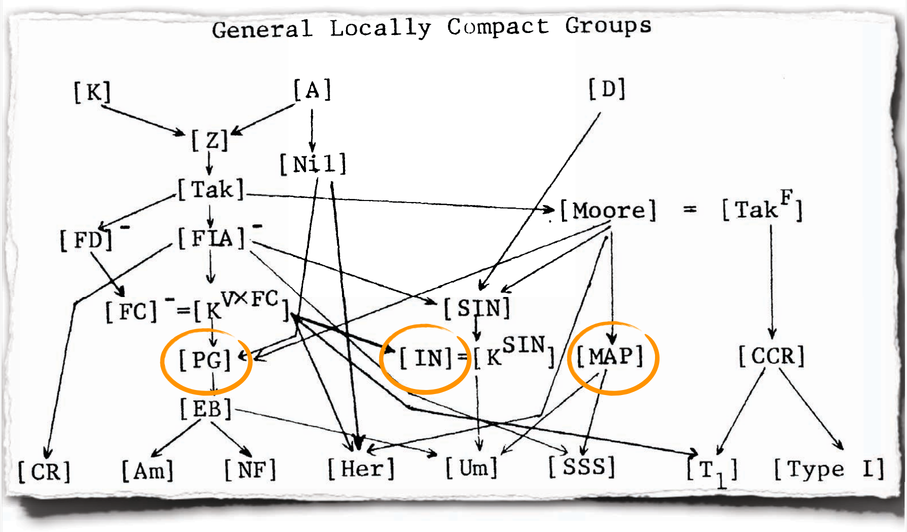
Theorem (J. Alaminos, J. Extremera, A. Villena (right now))

The derivation space $\mathrm{Der}(L^1(G))$ *is reflexive whenever* G *is a locally compact SSS-group.*

Open questions



Open questions



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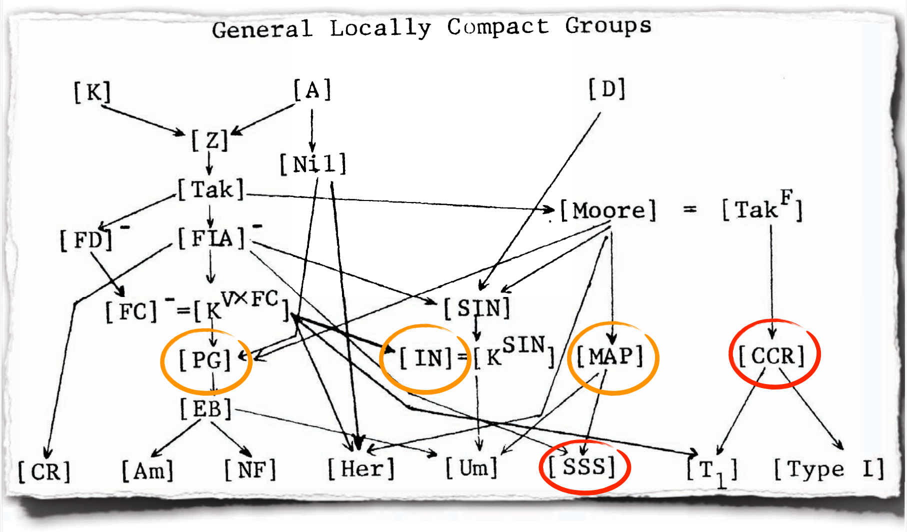
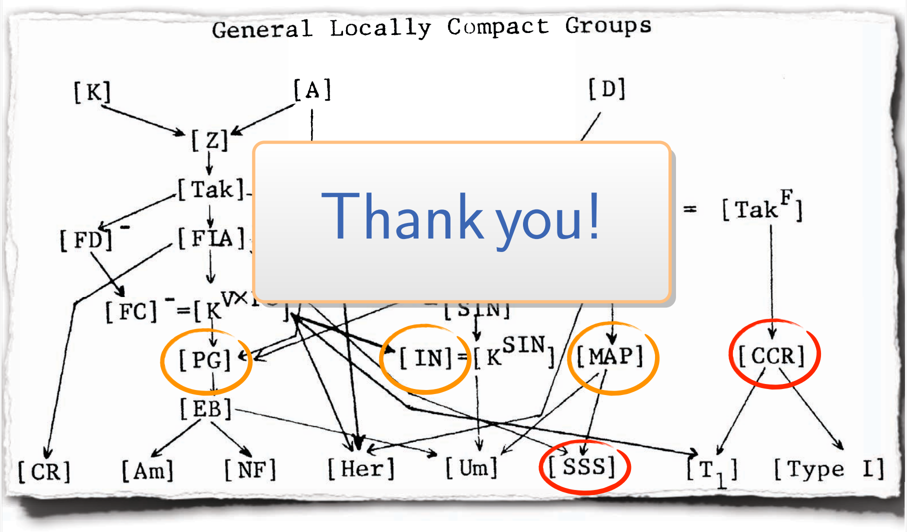


Open questions



Thank you!