Uniqueness of Power Series Representations

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Uniqueness of Power Series Representations

**Question:**

Can a function $f$ have more than one power series representation centered at $x = a$ with $R > 0$? That is, if

$$f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n$$

and

$$f(x) = \sum_{n=0}^{\infty} b_n (x - a)^n$$

must

$$a_n = b_n$$

for all $n \in \mathbb{N} \cup \{0\}$?
Theorem: [Differentiation of Power Series]

Assume that the power series \( \sum_{n=0}^{\infty} a_n (x - a)^n \) has radius of convergence \( R > 0 \). Let

\[
f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n
\]

for all \( x \in (a - R, a + R) \). Then \( f \) is differentiable on \( (a - R, a + R) \) and for each \( x \in (a - R, a + R) \),

\[
f'(x) = \sum_{n=1}^{\infty} n a_n (x - a)^{n-1}
\]
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**Observation:** If \( f(x) = \sum_{n=0}^{\infty} a_n(x - a)^n = \sum_{n=0}^{\infty} b_n(x - a)^n \), we have

\[
a_0 = f(a) = b_0
\]

Since \( f \) is differentiable on \((a - R, a + R)\) with

\[
f'(x) = \sum_{n=1}^{\infty} na_n(x - a)^{n-1} = a_1 + 2a_2(x - a) + 3a_3(x - a)^2 + \cdots
\]

and

\[
f'(x) = \sum_{n=1}^{\infty} nb_n(x - a)^{n-1} = b_1 + 2b_2(x - a) + 3b_3(x - a)^2 + \cdots
\]

we get that

\[
a_1 = f'(a) = b_1
\]

**Important Note:** \( f' \) is also differentiable on \((a - R, a + R)\).
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Observation (continued): Since

\[ f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-a)^{n-2} = 2a_2 + 3 \cdot 2a_3(x-a) + \cdots \]

and

\[ f''(x) = \sum_{n=2}^{\infty} n(n-1)b_n(x-a)^{n-2} = 2b_2 + 3 \cdot 2b_3(x-a) + \cdots \]

we get that

\[ 2a_2 = f''(a) = 2b_2 \]

Hence

\[ a_2 = \frac{f''(a)}{2} = b_2 \]
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**Observation (continued):** In fact $f$ has derivatives of all orders on $(a - R, a + R)$ with

$$f'''(x) = \sum_{n=3}^{\infty} n(n - 1)(n - 2)a_n(x - a)^{n-3}$$

and

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n - 1)(n - 2)(n - 3)\cdots(n - k + 1)a_n(x - a)^{n-k}$$

Hence

$$f'''(a) = 3!a_3 \Rightarrow a_3 = \frac{f'''(a)}{3!}$$

and

$$f^{(k)}(a) = k!a_k \Rightarrow a_k = \frac{f^{(k)}(a)}{k!}$$

**Note:** We would also have

$$b_k = \frac{f^{(k)}(a)}{k!}$$
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**Theorem: [Uniqueness of Power Series Representations]**

Suppose that
\[ f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n \]
for all \( x \in (a - R, a + R) \) where \( R > 0 \). Then
\[ a_n = \frac{f^{(n)}(a)}{n!} \]

In particular, if we also had that
\[ f(x) = \sum_{n=0}^{\infty} b_n (x - a)^n \]
then we must have that
\[ b_n = a_n \]
for each \( n = 0, 1, 2, 3, \ldots \)

**Remark:** \( f \) may have different representations at different center points.