Uniqueness of Power Series Representations

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Uniqueness of Power Series Representations

Question:

Can a function f have more than one power series representation centered at $x = a$ with $R > 0$? That is, if

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

and

$$f(x) = \sum_{n=0}^{\infty} b_n (x-a)^n$$

must

$$a_n = b_n$$

for all $n \in \mathbb{N} \cup \{0\}$?

Differentiation of Power Series

Theorem: [Differentiation of Power Series]

Assume that the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $R > 0$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for all $x \in (a-R, a+R)$. Then f is differentiable on $(a-R, a+R)$ and for each $x \in (a-R, a+R)$,

$$f'(x) = \sum_{n=1}^{\infty} n a_n(x-a)^{n-1}$$

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Observation: If $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = \sum_{n=0}^{\infty} b_n (x-a)^n$, we have

$$a_0 = f(a) = b_0$$

Since f is differentiable on $(a-R, a+R)$ with

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1} = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots$$

and

$$f'(x) = \sum_{n=1}^{\infty} n b_n (x-a)^{n-1} = b_1 + 2b_2(x-a) + 3b_3(x-a)^2 + \cdots$$

we get that

$$a_1 = f'(a) = b_1$$

Important Note: f' is also differentiable on $(a-R, a+R)$.

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Observation (continued): Since

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-a)^{n-2} = 2a_2 + 3\cdot 2a_3(x-a) + \cdots$$

and

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)b_n(x-a)^{n-2} = 2b_2 + 3\cdot 2b_3(x-a) + \cdots$$

we get that

$$2a_2 = f''(a) = 2b_2$$

Hence

$$a_2 = \frac{f''(a)}{2} = b_2$$

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Observation (continued): In fact f has derivatives of all orders on $(a-R, a+R)$ with

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)a_n(x-a)^{n-3}$$

and

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)(n-3)\cdots(n-k+1)a_n(x-a)^{n-k}$$

Hence

$$f'''(a) = 3!a_3 \Rightarrow a_3 = \frac{f'''(a)}{3!}$$

and

$$f^{(k)}(a) = k!a_k \Rightarrow a_k = \frac{f^{(k)}(a)}{k!}$$

Note: We would also have

$$b_k = \frac{f^{(k)}(a)}{k!}$$

Uniqueness of Power Series Representations

Theorem: [Uniqueness of Power Series Representations]

Suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

for all $x \in (a-R, a+R)$ where $R > 0$. Then

$$a_n = \frac{f^{(n)}(a)}{n!}$$

In particular, if we also had that

$$f(x) = \sum_{n=0}^{\infty} b_n (x-a)^n$$

then we must have that

$$b_n = a_n$$

for each $n = 0, 1, 2, 3, \cdots$

Remark: f may have different representations at different center points.