Taylor Polynomials: Examples

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Taylor Polynomials

Recall:

Definition: [Taylor Polynomials]

Assume that $f(x)$ is n-times differentiable at $x = a$. The *n-th degree Taylor polynomial for $f(x)$ centered at $x = a$* is the polynomial

$$\begin{aligned} T_{n,a}(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \\ &\quad \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

Observation: Using the convention where $0! = 1! = 1$ and $(x-a)^0 = 1$, we have the following:

$$\begin{aligned} T_{0,a}(x) &= \frac{f(a)}{0!}(x-a)^0 = f(a) \\ T_{1,a}(x) &= \frac{f(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 = f(a) + f'(a)(x-a) \\ &= L_a^f(x). \end{aligned}$$

Taylor Polynomials for $\cos(x)$

Example 1: Find all of the Taylor polynomials up to degree 5 for the function $f(x) = \cos(x)$ with center $x = 0$.

Solution: We know that

$$\begin{aligned} f(0) &= \cos(0) = 1, \\ f'(0) &= -\sin(0) = 0, \text{ and} \\ f''(0) &= -\cos(0) = -1. \end{aligned}$$

It follows that

$$\begin{aligned} T_{0,0}(x) &= 1, \\ T_{1,0}(x) &= L_0(x) = 1 + 0(x-0) = 1, \text{ and} \\ T_{2,0}(x) &= 1 + 0(x-0) + \frac{-1}{2!}(x-0)^2 = 1 - \frac{x^2}{2} \end{aligned}$$

for all x.

Note:

$$T_{0,0}(x) = 1 = T_{1,0}(x).$$

Taylor Polynomials for $\cos(x)$

Example 1 (continued): Find all of the Taylor polynomials up to degree 5 for the function $f(x) = \cos(x)$ with center $x = 0$.

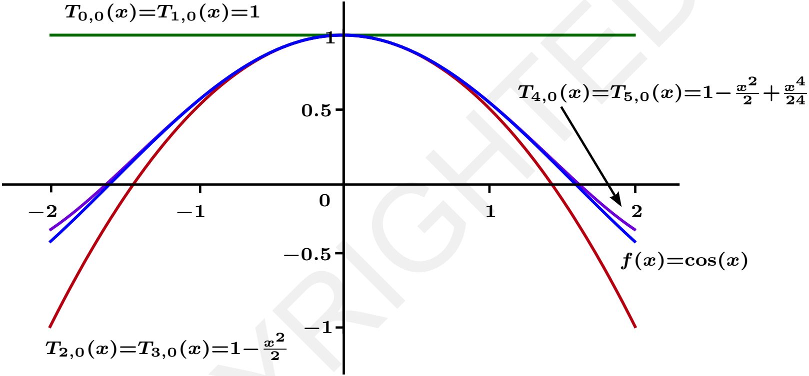
Solution (continued): Recall $f'''(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$, and $f^{(5)}(x) = -\sin(x)$, we get $f'''(0) = \sin(0) = 0$, $f^{(4)}(0) = \cos(0) = 1$ and $f^{(5)}(0) = -\sin(0) = 0$. Hence,

$$\begin{aligned} T_{3,0}(x) &= 1 + 0(x-0) + \frac{-1}{2!}(x-0)^2 + \frac{0}{3!}(x-0)^3 \\ &= 1 - \frac{x^2}{2} \\ &= T_{2,0}(x) \end{aligned}$$

We also have that

$$\begin{aligned} T_{4,0}(x) &= 1 + 0(x-0) + \frac{-1}{2!}(x-0)^2 + \frac{0}{3!}(x-0)^3 + \frac{1}{4!}(x-0)^4 \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{24} \\ &= T_{5,0}(x) \end{aligned}$$

Taylor Polynomials for $\cos(x)$



Note: The diagram displays the graph of $\cos(x)$ with its Taylor Polynomials up to degree 5.

For $k \geq 0$

$$T_{2k,0}(x) = T_{2k+1,0}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + (-1)^k \frac{x^{2k}}{2k!}$$

Taylor Polynomials for $\sin(x)$

Example 2: Find all of the Taylor polynomials up to degree 5 for the function $f(x) = \sin(x)$ with center $x = 0$.

Solution : We can see that

$$\begin{aligned}
f(0) &= \sin(0) = 0, \\
f'(0) &= \cos(0) = 1, \\
f''(0) &= -\sin(0) = 0, \\
f'''(0) &= -\cos(0) = -1, \\
f^{(4)}(0) &= \sin(0) = 0, \text{ and} \\
f^{(5)}(0) &= \cos(0) = 1.
\end{aligned}$$

Therefore

$$\begin{aligned}
T_{0,0}(x) &= 0, \\
T_{1,0}(x) &= L_0(x) = 0 + 1(x - 0) = x
\end{aligned}$$

and

$$\begin{aligned}
T_{2,0}(x) &= 0 + 1(x - 0) + \frac{0}{2!}(x - 0)^2 \\
&= x \\
&= T_{1,0}(x).
\end{aligned}$$

Taylor Polynomials for $\sin(x)$

Example 2 (continued): Find all of the Taylor polynomials up to degree 5 for the function $f(x) = \sin(x)$ with center $x = 0$.

Solution (continued): Recall that $f(0) = \sin(0) = 0$, $f'(0) = \cos(0) = 1$, $f''(0) = -\sin(0) = 0$, $f'''(0) = -\cos(0) = -1$, $f^{(4)}(0) = \sin(0) = 0$, and $f^{(5)}(0) = \cos(0) = 1$.

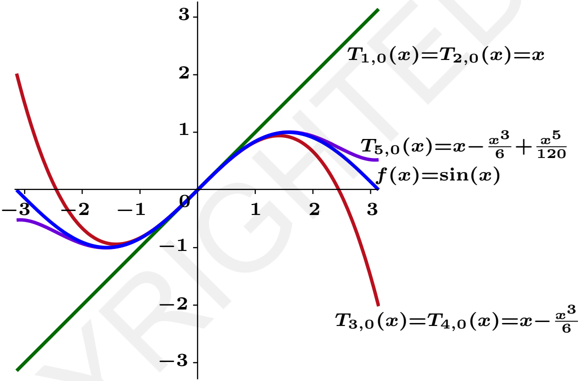
Next we have

$$\begin{aligned} T_{3,0}(x) &= 0 + 1(x-0) + \frac{0}{2!}(x-0)^2 + \frac{-1}{3!}(x-0)^3 \\ &= x - \frac{x^3}{6} \\ &= T_{4,0}(x) \end{aligned}$$

Finally,

$$T_{5,0}(x) = T_{4,0}(x) + \frac{x^5}{5!} = x - \frac{x^3}{6} + \frac{x^5}{120}$$

Taylor Polynomials for $\sin(x)$



Note: The diagram displays the graph of $\sin(x)$ with its Taylor Polynomials up to degree 5 (excluding $T_{0,0}(x)$ since its graph is the x-axis).

For $k \geq 0$

$$T_{2k+1,0}(x) = T_{2k+2,0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{k-1}\frac{x^{2k+1}}{(2k+1)!}$$

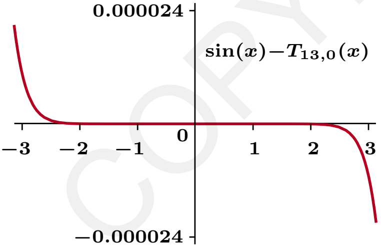
Taylor Polynomials for $\sin(x)$

Question: How accurate is the approximation

$$\sin(x) \cong T_{13,0}(x)$$

where

$$\begin{aligned} T_{13,0}(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 \\ &\quad + \frac{1}{362880}x^9 - \frac{1}{39916800}x^{11} + \frac{1}{6227020800}x^{13} ? \end{aligned}$$



Note: If $x \in [-1, 1]$, then

$$|\sin(x) - T_{13,0}(x)| < 10^{-12}$$

while for $x \in [-0.01, 0.01]$,

$$|\sin(x) - T_{13,0}(x)| < 10^{-42}.$$

Taylor Polynomials for e^x

Example: Let $f(x) = e^x$. Then

$$f^{(k)}(x) = e^x \Rightarrow f^{(k)}(0) = e^0 = 1$$

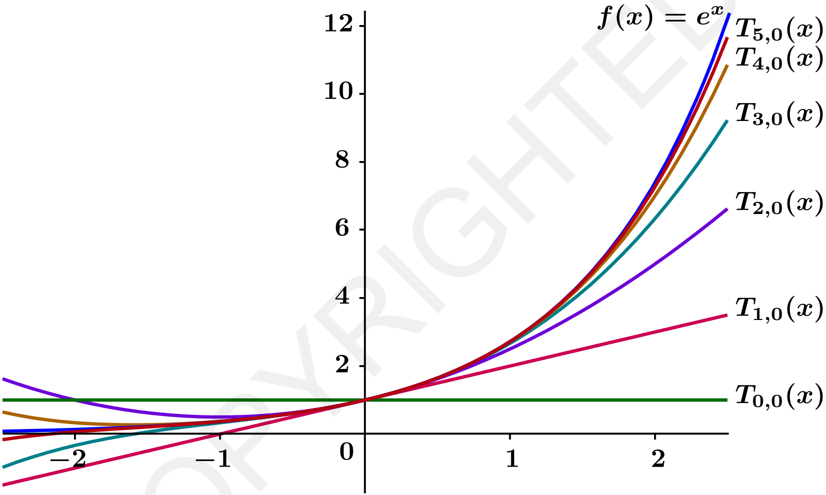
for any k. Therefore

$$\begin{aligned} T_{n,0}(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k \\ &= \sum_{k=0}^{n} \frac{e^0}{k!} x^k \\ &= \sum_{k=0}^{n} \frac{x^k}{k!}. \end{aligned}$$

In particular,

$$\begin{aligned} T_{0,0}(x) &= 1, \\ T_{1,0}(x) &= 1 + x, \\ T_{2,0}(x) &= 1 + x + \frac{x^2}{2}, \\ T_{3,0}(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}, \\ T_{4,0}(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}, \\ T_{5,0}(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}. \end{aligned}$$

Taylor Polynomials for e^x



$$T_{n,0}(x) = \sum_{k=0}^{n} \frac{x^{(k)}}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$