Taylor Polynomials: Examples

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Taylor Polynomials

Recall:

**Definition: [Taylor Polynomials]**

Assume that $f(x)$ is $n$-times differentiable at $x = a$. The $n$-th degree Taylor polynomial for $f(x)$ centered at $x = a$ is the polynomial

$$T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k$$

$$= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

**Observation:** Using the convention where $0! = 1! = 1$ and $(x - a)^0 = 1$, we have the following:

$$T_{0,a}(x) = \frac{f(a)}{0!}(x - a)^0 = f(a)$$

$$T_{1,a}(x) = \frac{f(a)}{0!}(x - a)^0 + \frac{f'(a)}{1!}(x - a)^1 = f(a) + f'(a)(x - a) = L_a^f(x).$$
Taylor Polynomials for $\cos(x)$

**Example 1:** Find all of the Taylor polynomials up to degree 5 for the function $f(x) = \cos(x)$ with center $x = 0$.

**Solution:** We know that

\[
\begin{align*}
    f(0) &= \cos(0) = 1, \\
    f'(0) &= -\sin(0) = 0, \text{ and} \\
    f''(0) &= -\cos(0) = -1.
\end{align*}
\]

It follows that

\[
\begin{align*}
    T_{0,0}(x) &= 1, \\
    T_{1,0}(x) &= L_0(x) = 1 + 0(x - 0) = 1, \text{ and} \\
    T_{2,0}(x) &= 1 + 0(x - 0) + \frac{-1}{2!}(x - 0)^2 = 1 - \frac{x^2}{2}
\end{align*}
\]

for all $x$.

**Note:**

$$T_{0,0}(x) = 1 = T_{1,0}(x).$$
Taylor Polynomials for $\cos(x)$

Example 1 (continued): Find all of the Taylor polynomials up to degree 5 for the function $f(x) = \cos(x)$ with center $x = 0$.

Solution (continued): Recall $f'''(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$, and $f^{(5)}(x) = -\sin(x)$, we get $f'''(0) = \sin(0) = 0$, $f^{(4)}(0) = \cos(0) = 1$ and $f^{(5)}(0) = -\sin(0) = 0$. Hence,

$$T_{3,0}(x) = 1 + 0(x - 0) + \frac{-1}{2!}(x - 0)^2 + \frac{0}{3!}(x - 0)^3$$

$$= 1 - \frac{x^2}{2}$$

$$= T_{2,0}(x)$$

We also have that

$$T_{4,0}(x) = 1 + 0(x - 0) + \frac{-1}{2!}(x - 0)^2 + \frac{0}{3!}(x - 0)^3 + \frac{1}{4!}(x - 0)^4$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$= T_{5,0}(x)$$
Taylor Polynomials for $\cos(x)$

\[ T_{0,0}(x) = T_{1,0}(x) = 1 \]

\[ T_{2,0}(x) = T_{3,0}(x) = 1 - \frac{x^2}{2} \]

\[ T_{4,0}(x) = T_{5,0}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} \]

\[ f(x) = \cos(x) \]

\[ T_{2k,0}(x) = T_{2k+1,0}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + (-1)^k \frac{x^{2k}}{2k!} \]

**Note:** The diagram displays the graph of $\cos(x)$ with its Taylor Polynomials up to degree 5.

For $k \geq 0$
Taylor Polynomials for \( \sin(x) \)

**Example 2:** Find all of the Taylor polynomials up to degree 5 for the function \( f(x) = \sin(x) \) with center \( x = 0 \).

**Solution:** We can see that

\[
\begin{align*}
f(0) &= \sin(0) = 0, \\
f'(0) &= \cos(0) = 1, \\
f''(0) &= -\sin(0) = 0, \\
f'''(0) &= -\cos(0) = -1, \\
f^{(4)}(0) &= \sin(0) = 0, \text{ and} \\
f^{(5)}(0) &= \cos(0) = 1.
\end{align*}
\]

Therefore

\[
\begin{align*}
T_{0,0}(x) &= 0, \\
T_{1,0}(x) &= L_0(x) = 0 + 1(x - 0) = x \\
T_{2,0}(x) &= 0 + 1(x - 0) + \frac{0}{2!}(x - 0)^2 \\
&= x \\
&= T_{1,0}(x).
\end{align*}
\]
Taylor Polynomials for $\sin(x)$

Example 2 (continued): Find all of the Taylor polynomials up to degree 5 for the function $f(x) = \sin(x)$ with center $x = 0$.

Solution (continued): Recall that $f(0) = \sin(0) = 0$, $f'(0) = \cos(0) = 1$, $f''(0) = -\sin(0) = 0$, $f'''(0) = -\cos(0) = -1$, $f^{(4)}(0) = \sin(0) = 0$, and $f^{(5)}(0) = \cos(0) = 1$.

Next we have

$$T_{3,0}(x) = 0 + 1(x - 0) + \frac{0}{2!}(x - 0)^2 + \frac{-1}{3!}(x - 0)^3$$

$$= x - \frac{x^3}{6}$$

$$= T_{4,0}(x)$$

Finally,

$$T_{5,0}(x) = T_{4,0}(x) + \frac{x^5}{5!} = x - \frac{x^3}{6} + \frac{x^5}{120}$$
Taylor Polynomials for $\sin(x)$

Note: The diagram displays the graph of $\sin(x)$ with its Taylor Polynomials up to degree 5 (excluding $T_{0,0}(x)$ since its graph is the $x$-axis).

For $k \geq 0$

$$T_{2k+1,0}(x) = T_{2k+2,0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{k-1} \frac{x^{2k+1}}{(2k+1)!}$$
Taylor Polynomials for $\sin(x)$

**Question:** How accurate is the approximation

$$\sin(x) \approx T_{13,0}(x)$$

where

$$T_{13,0}(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \frac{1}{39916800}x^{11} + \frac{1}{6227020800}x^{13}?$$

**Note:** If $x \in [-1, 1]$, then

$$| \sin(x) - T_{13,0}(x) | < 10^{-12}$$

while for $x \in [-0.01, 0.01]$,

$$| \sin(x) - T_{13,0}(x) | < 10^{-42}.$$
Taylor Polynomials for $e^x$

**Example:** Let $f(x) = e^x$. Then

\[ f^{(k)}(x) = e^x \Rightarrow f^{(k)}(0) = e^0 = 1 \]

for any $k$. Therefore

\[
T_{n,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} (x - 0)^k
\]

\[
= \sum_{k=0}^{n} \frac{e^0}{k!} x^k
\]

\[
= \sum_{k=0}^{n} \frac{x^k}{k!}.
\]

In particular,

\[
T_{0,0}(x) = 1,
\]

\[
T_{1,0}(x) = 1 + x,
\]

\[
T_{2,0}(x) = 1 + x + \frac{x^2}{2},
\]

\[
T_{3,0}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6},
\]

\[
T_{4,0}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24},
\]

\[
T_{5,0}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}.
\]
Taylor Polynomials for $e^x$

$f(x) = e^x$

$T_n,0(x) = \sum_{k=0}^{n} \frac{x^{(k)}}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$