Finding the Radius of Convergence

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Interval and Radius of Convergence

Definition: [Interval and Radius of Convergence]

Given a power series of the form $\sum_{n=0}^{\infty} a_n(x-a)^n$, the set

$$I = \{x_0 \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n(x_0 - a)^n \text{ converges}\}$$

is an interval centered at $x = a$ which we call *the interval of convergence* for the power series.

Let

$$R = \begin{cases} lub(\{|x_0 - a| \mid x_0 \in I\}) & \text{if } I \text{ is bounded,} \\ \infty & \text{if } I \text{ is not bounded.} \end{cases}$$

Then R is called the *radius of convergence* of the power series.

Radius of Convergence

Theorem: [Fundamental Convergence Theorem for Power Series]

Given a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ centered at $x=a$, let R be the radius of convergence.

1. If $R=0$, then $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges for $x=a$, but it diverges for all other values of x.
2. If $0<R<\infty$, then the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely for every $x \in (a-R, a+R)$ and diverges if $\mid x-a \mid > R$.
3. If $R=\infty$, then the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely for every $x \in \mathbb{R}$.

Radius of Convergence

Question: How do we find the radius of convergence R?

Key Observation: Given $\sum_{n=0}^{\infty} a_n x^n$, assume that

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

where $0 \leq L < \infty$. For $x_0 \neq 0$, let

$$b_n = a_n x_0^n$$

then

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| &= \lim_{n\to\infty} \left| \frac{a_{n+1} x_0^{n+1}}{a_n x_0^n} \right| \\
&= \lim_{n\to\infty} |x_0| \left| \frac{a_{n+1}}{a_n} \right| \\
&= |\, x_0 \,| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= L \,|\, x_0 \,|
\end{aligned}$$

Radius of Convergence

Conclusions:

1. The Ratio Test shows that the series $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_n x_0^n$ converges absolutely if $L \mid x_0 \mid < 1$ and diverges if $L \mid x_0 \mid > 1$.
2. Assume that $0 < L < \infty$. Then $L \mid x_0 \mid < 1$ if and only if $\mid x_0 \mid < \frac{1}{L}$. Therefore, the radius of convergence is $\frac{1}{L}$.
3. If $L = 0$, then no matter the value of x_0, we have $L \mid x_0 \mid = 0 < 1$. Therefore, $R = \infty$.
4. If

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty,$$

then the same calculation would show that

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \infty$$

and as such the series diverges for all nonzero x_0. But we know that the series must converge at $x = 0$, so $R = 0$.

Radius of Convergence

Theorem: [Test for the Radius of Convergence]

Let $\sum_{n=0}^{\infty} a_n(x-a)^n$ be a power series for which

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

where $0 \leq L < \infty$ or $L = \infty$. Let R be the radius of convergence of the power series.

1. If $0 < L < \infty$, then $R = \frac{1}{L}$.
2. If $L = 0$, then $R = \infty$.
3. If $L = \infty$, then $R = 0$.

Example

Example: Find the radius and interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n(n^2+1)}$$

Solution: Let $a_n = \dfrac{1}{3^n(n^2+1)}$. Then

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{1}{3^{n+1}((n+1)^2+1)}}{\frac{1}{3^n(n^2+1)}}\right| \\
&= \lim_{n\to\infty}\left|\frac{3^n(n^2+1)}{3^{n+1}((n+1)^2+1)}\right| \\
&= \lim_{n\to\infty}\frac{1}{3}\left(\frac{n^2+1}{n^2+2n+2}\right) \\
&= \frac{1}{3}\lim_{n\to\infty}\frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}} \\
&= \frac{1}{3}
\end{aligned}$$

It follows from the Radius of Convergence Test that $R = \frac{1}{\frac{1}{3}} = 3$.

Example

Example (continued):

For $x = 3$, the series becomes

$$\sum_{n=0}^{\infty} \frac{3^n}{3^n(n^2+1)} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}$$

Since

$$\frac{1}{n^2+1} < \frac{1}{n^2}$$

and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the Comparison Test shows that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges and hence that

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1}$$

converges.

Example

Example (continued):

Similarly, if $x = -3$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n(n^2+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$$

Then the Alternating Series Test applies and this series also converges.

Alternately, we have

$$\left| \frac{(-1)^n}{n^2+1} \right| = \frac{1}{n^2+1}$$

and since $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges, this shows $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ converges absolutely.

Therefore, the interval of convergence is $[-3, 3]$.

Radius of Convergence

Key Observation: The series

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n(n^2+1)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{x^n}{3^n}$$

have the same radius of convergence!

Theorem

Let p and q be non-zero polynomials where $q(n) \neq 0$ for $n \geq k$. Then the following series have the same radius of convergence:

1. $\sum_{n=k}^{\infty} a_n(x-a)^n$
2. $\sum_{n=k}^{\infty} \frac{a_n p(n)(x-a)^n}{q(n)}$

However, they may have different intervals of convergence.

Radius of Convergence

Example: Find the radius and interval of convergence for the series

$$1 + 2x + x^2 + 2x^3 + x^4 + \cdots$$

In this case

$$\frac{a_{n+1}}{a_n} = \begin{cases} 2 & \text{if } n \text{ is even} \\ \frac{1}{2} & \text{if } n \text{ is odd} \end{cases}$$

so

$$\lim_{n\to\infty} \frac{|\, a_{n+1} \,|}{|\, a_n \,|}$$

does not exist.

Observations:

1) The Divergence Test shows the series diverges if $|\, x \,| \geq 1$.
2) If $0 \leq x_0 < 1$, then

   $$0 < 1+2x_0+x_0^2+2x_0^3+x_0^4+\cdots \leq 2(1+x_0+x_0^2+x_0^3+x_0^4+\cdots)$$

   so the series converges by the Comparison Test and the Geometric Series Test.

   Hence $R = 1$ and the interval of convergence is $(-1, 1)$.