More Examples of Power Series

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Example 1: Using the Generalized Binomial Theorem we saw that

$$(1+x)^{-2} = \sum_{k=1}^{\infty} (-1)^{k-1} k x^{k-1}$$

Verify this using term-by-term differentiation.

Solution: We know that for $u \in (-1, 1)$

$$\frac{1}{1-u} = \sum_{k=0}^{\infty} u^k$$

so with $u = -x$,

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$$

Term-by-term differentiation gives

$$-\frac{1}{(1+x)^2} = \sum_{k=1}^{\infty} (-1)^k k x^{k-1}$$

Factoring out -1 gives

$$\frac{1}{(1+x)^2} = \sum_{k=1}^{\infty} (-1)^{k-1} k x^{k-1}$$

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Example 2: We know that $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ for $u \in (-1, 1)$. Hence, if we let $u = -x^2$, then

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

for all $x \in (-1, 1)$.

It follows that

$$\arctan(x) = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

for some C. But $\arctan(0) = 0 \Rightarrow C = 0$ and

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Moreover, because $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ also converges at $x = 1$, we have

$$\frac{\pi}{4} = \arctan(1) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \Rightarrow \pi = \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1}$$

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Example 3:

i) Find the Taylor series centered at $x = 0$ for the integral function

$$F(x) = \int_0^x \cos(t^2)\,dt$$

ii) Find $F^{(9)}(0)$ and $F^{(16)}(0)$.

iii) Estimate $\int_0^{0.1} \cos(t^2)\,dt$ with an error of less than $\frac{1}{10^6}$.

Solution: i) We know that for any $u \in \mathbb{R}$,

$$\cos(u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!}$$

If we let $u = t^2$, we get that for any $t \in \mathbb{R}$,

$$\cos(t^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(t^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n}}{(2n)!}$$

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Example 3 (continued): The Integration Theorem for Power Series gives us that

$$\begin{aligned} F(x) &= \int_0^x \cos(t^2)\, dt \\ &= \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n}}{(2n)!}\, dt \\ &= \sum_{n=0}^{\infty} \int_0^x (-1)^n \frac{t^{4n}}{(2n)!}\, dt \\ &= \sum_{n=0}^{\infty} \left[(-1)^n \, \frac{t^{4n+1}}{(4n+1)(2n)!} \bigg|_0^x \right] \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!} \end{aligned}$$

for all $x \in \mathbb{R}$.

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Example 3 (continued): ii) We have

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!}$$

We know that

$$F^{(9)}(0) = a_9 9! \quad \text{and} \quad F^{(16)}(0) = a_{16} 16!$$

Then

$$4n+1 = 9 \Rightarrow n = 2$$

so that

$$a_9 = (-1)^2 \frac{1}{(4(2)+1)(2 \cdot 2)!} = \frac{1}{9 \cdot 4!}$$

Hence

$$F^{(9)}(0) = \frac{1}{9 \cdot 4!} \cdot 9! = 5 \cdot 6 \cdot 7 \cdot 8 = 1680$$

Since $4n+1 \neq 16$ for any $n \in \mathbb{N} \cup \{0\}$, $a_{16} = 0$ and

$$F^{(16)}(0) = 0$$

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Example 3 (continued): iii) Estimate $\int_0^{0.1} \cos(t^2)\,dt$ with an error of less than $\frac{1}{10^6}$.

Since

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!}$$

we have

$$\int_0^{0.1} \cos(t^2)\,dt = F(0.1) = \sum_{n=0}^{\infty} (-1)^n \frac{(0.1)^{4n+1}}{(4n+1)(2n)!}$$

This is an alternating series with

$$a_n = \frac{(0.1)^{4n+1}}{(4n+1)(2n)!}$$

Then

$$a_1 = \frac{(0.1)^5}{(5)(2)!} = \frac{1}{10^6}$$

and $\sum_{n=0}^{0} (-1)^n \frac{(0.1)^{4n+1}}{(4n+1)(2n)!} = (-1)^0 \frac{(0.1)}{(1)(0!)} = 0.1 = a_0$ so

$$\left| \int_0^{0.1} \cos(t^2)\,dt - 0.1 \right| < a_1 = \frac{1}{10^6}$$