Introduction to Power Series

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Example

Problem: For which values of $x$ does the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converge?

Solution: Observe that if $b_n = | \frac{x^n}{n} |$ then

$$\lim_{n \to \infty} \frac{b_{n+1}}{b_n} = \lim_{n \to \infty} \frac{|x|^{n+1}}{n+1} \cdot \frac{n}{|x|^n}$$

$$= \lim_{n \to \infty} |x| \cdot \frac{n}{n+1}$$

$$= |x|$$

The Ratio Test shows that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges absolutely if $|x| < 1$ and the series diverges if $|x| > 1$.

If $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

If $x = -1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges.
**Definition: [Power Series]**

A *power series centered at* $x = a$ is a formal series of the form

$$
\sum_{n=0}^{\infty} a_n (x - a)^n
$$

where $x$ is viewed as a variable.

The value $a_n$ is called the *coefficient* of the term $(x - a)^n$.

**Central Questions:**

1) If $I = \{x_0 \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n (x_0 - a)^n$ converges}, then what can we say about $I$?

2) If we define a function $f$ on $I$ by $f(x_0) = \sum_{n=0}^{\infty} a_n (x_0 - a)^n$ for all $x_0 \in I$, then what can we say about $f(x)$?
Power Series

Observations:

1) Our convention is that $0^0 = 1$, so that if $x = a$, the series becomes

$$\sum_{n=0}^{\infty} a_n (a - a)^n = a_0 + 0 + 0 + 0 + \cdots = a_0$$

2) We can let $u = x - a$ to get that $\sum_{n=0}^{\infty} a_n (x - a)^n$ converges at $x = x_0$ if and only if $\sum_{n=0}^{\infty} a_n u^n$ converges at $u = x_0 - a$.

Consequence: We need only focus on series of the form

$$\sum_{n=0}^{\infty} a_n x^n$$
Interval of Convergence

Note: The series \( \sum_{n=0}^{\infty} a_n x^n \)
always converges at \( x = 0 \).
Assume that \( \sum_{n=0}^{\infty} a_n x_1^n \)
converges where \( x_1 \neq 0 \).

Then \( \lim_{n \to \infty} | a_n x_1^n | = 0 \) and there exists an \( N_0 \in \mathbb{N} \) so that if \( n \geq N_0 \) we have \( | a_n x_1^n | \leq 1 \).

Let \( | x_0 | < | x_1 | \). Then if \( n \geq N_0 \)

\[
| a_n x_0^n | = | a_n x_1^n | | \frac{x_0}{x_1} |^n \leq | \frac{x_0}{x_1} |^n
\]

Since \( | \frac{x_0}{x_1} | < 1 \), the series

\[
\sum_{n=N_0}^{\infty} | \frac{x_0}{x_1} |^n
\]

converges. By the Comparison Test we have that

\[
\sum_{n=0}^{\infty} a_n x_0^n \]
converges absolutely.
Interval of Convergence

**Key Observations:** If the power series $ \sum_{n=0}^{\infty} a_n x^n$ converges at $x_1 \neq 0$ and if $|x_0| < |x_1|$, then the series converges at $x_0$ as well.

Let

$$I = \{x_0 \in \mathbb{R} | \sum_{n=0}^{\infty} a_n x_0^n \text{ converge}\}.$$

Then

$$(-|x_1|, |x_1|) \subset I$$

$\Rightarrow I$ is an interval centered around $x = 0$.

**Note:** The same analysis works for power series of the form $\sum_{n=0}^{\infty} a_n (x - a)^n$. 
Definition: [Interval and Radius of Convergence]

Given a power series of the form \( \sum_{n=0}^{\infty} a_n(x - a)^n \), the set

\[
I = \{ x_0 \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n(x_0 - a)^n \text{ converges} \}
\]

is an interval centered at \( x = a \) which we call the interval of convergence for the power series.

Let

\[
R = \begin{cases} 
\text{lub}(\{|x_0 - a| \mid x_0 \in I\}) & \text{if } I \text{ is bounded}, \\
\infty & \text{if } I \text{ is not bounded}.
\end{cases}
\]

Then \( R \) is called the radius of convergence of the power series.
Theorem: [Fundamental Convergence Theorem for Power Series]

Given a power series \( \sum_{n=0}^{\infty} a_n (x - a)^n \) centered at \( x = a \), let \( R \) be the radius of convergence.

1. If \( R = 0 \), then \( \sum_{n=0}^{\infty} a_n (x - a)^n \) converges for \( x = a \), but it diverges for all other values of \( x \).

2. If \( 0 < R < \infty \), then the series \( \sum_{n=0}^{\infty} a_n (x - a)^n \) converges absolutely for every \( x \in (a - R, a + R) \) and diverges if \( |x - a| > R \).

3. If \( R = \infty \), then the series \( \sum_{n=0}^{\infty} a_n (x - a)^n \) converges absolutely for every \( x \in \mathbb{R} \).
Interval of Convergence

**Remark:** If $0 < R < \infty$, then there are four possibilities for the interval of convergence $I$.

1) $I = (a - R, a + R)$  
   Example: $\sum_{n=0}^{\infty} x^n \Rightarrow I = (-1, 1)$.

2) $I = [a - R, a + R)$  
   Example: $\sum_{n=1}^{\infty} \frac{x^n}{n} \Rightarrow I = [-1, 1)$.

3) $I = (a - R, a + R]$  
   Example: $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \Rightarrow I = (-1, 1]$.

4) $I = [a - R, a + R]$  
   Example: $\sum_{n=1}^{\infty} \frac{x^n}{n^2} \Rightarrow I = [-1, 1]$.

**Key Note:** Once $R$ is determined, you need to test the endpoints separately.