Differentiation of Power Series

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Functions Represented by Power Series

Question: Assume that the series \( \sum_{n=0}^{\infty} a_n (x - a)^n \) has radius of convergence \( R > 0 \) and interval of convergence \( I \). What are the properties of the function

\[
f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n \]

1. Is \( f \) continuous on \( I \)?

Theorem: [Abel’s Theorem: Continuity of Power Series]

Assume that the power series \( \sum_{n=0}^{\infty} a_n (x - a)^n \) has interval of convergence \( I \). Let

\[
f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n
\]

for each \( x \in I \). Then \( f \) is continuous on \( I \).

2. Is \( f \) differentiable on \( (a - R, a + R) \)?
Differentiation of Power Series

**Strategy:** If we have a function

\[ f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n \]

that is represented by a power series with radius of convergence \( R > 0 \), we could try to differentiate \( f \) by differentiating the series one term at a time.

Since \( \frac{d}{dx} (a_n (x - a)^n) = n a_n (x - a)^{n-1} \), we get:

**Definition: [Formal Derivative of a Power Series]**

Given a power series \( \sum_{n=0}^{\infty} a_n (x - a)^n \), the *formal derivative* is the series

\[ \sum_{n=1}^{\infty} n a_n (x - a)^{n-1} \]
Differentiation of Power Series

Two Fundamental Problems:

Problem 1: For which values of $x$ does the formal power series

$$\sum_{n=1}^{\infty} na_n (x - a)^{n-1}$$

converge? In particular, does this series converge for the same values as the original series $\sum_{n=0}^{\infty} a_n (x - a)^n$?

Problem 2: If both of the series $\sum_{n=0}^{\infty} a_n (x - a)^n$ and $\sum_{n=1}^{\infty} na_n (x - a)^{n-1}$ converge at the same $x$, must it be the case that

$$f'(x) = \sum_{n=1}^{\infty} na_n (x - a)^{n-1}?$$
Differentiation of Power Series

**Problem 1:** For which values of $x$ does the formal power series

$$\sum_{n=1}^{\infty} na_n(x - a)^{n-1}$$

converge? In particular, does this series converge for the same values as the original series $\sum_{n=0}^{\infty} a_n(x - a)^n$?

**Observation:** The series $\sum_{n=0}^{\infty} a_n(x - a)^n$ and the series $\sum_{n=0}^{\infty} na_n(x - a)^n$ have the same radius of convergence.

We can show that the series $\sum_{n=0}^{\infty} a_n(x - a)^n$ and its formal derivative

$$\sum_{n=1}^{\infty} na_n(x - a)^{n-1}$$

also have the same radius of convergence, though the interval of convergence may be different. Therefore,

$$g(x) = \sum_{n=1}^{\infty} na_n(x - a)^{n-1}$$

is defined for all $x \in (a - R, a + R)$. Is $g(x) = f'(x)$?
Theorem: [Differentiation of Power Series]

Assume that the power series \( \sum_{n=0}^{\infty} a_n (x - a)^n \) has radius of convergence \( R > 0 \). Let

\[
f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n
\]

for all \( x \in (a - R, a + R) \). Then \( f \) is differentiable on \( (a - R, a + R) \) and for each \( x \in (a - R, a + R) \),

\[
f'(x) = \sum_{n=1}^{\infty} n a_n (x - a)^{n-1}
\]
Differentiation of Power Series

**Example:** If $|x| < 1$, then let

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Differentiating term-by-term, we get

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

**Question:** Evaluate

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$$

**Observation:** This series is obtained from $\sum_{n=1}^{\infty} nx^{n-1}$ by letting $x = \frac{1}{2}$. Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = f'(\frac{1}{2}) = \frac{1}{(1-\frac{1}{2})^2} = 4$$
Differentiation of Power Series

**Example:** For any $x \in \mathbb{R}$ let

$$g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Term-by-term differentiation gives

$$g'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= g(x)$$

Hence

$$g(x) = C e^x$$

for some constant $C$. However, $C = g(0) = 1$, so

$$g(x) = e^x$$
Differentiation of Power Series

Example: Find a power series representation for the function

\[ f(x) = e^{-x^2} \]

We have that for any \( u \in \mathbb{R} \) that

\[ e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \quad (\star) \]

Let \( u = -x^2 \) and substitute for \( u \) in the expression \((\star)\) to get

\[ e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \]

for all \( x \in \mathbb{R} \).

Note: It may look like

\[ e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots \]

is not a power series since there are no terms involving \( x^n \) when \( n \) is odd. But in fact, it really is a power series where the coefficients are of the form \( a_{2k-1} = 0 \) and \( a_{2k} = (-1)^k \frac{1}{(k)!} \) for each \( k = 0, 1, 2, 3, 4, \ldots \)
A Strange Function

**Question:** Why are power series so special?

**Example:** Let

\[ f(x) = \sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^n \sin(9^n x) \]

for all \( x \in \mathbb{R} \).

**Fact:** The function \( f \) is continuous on \( \mathbb{R} \) but it is not differentiable at a single point.