Building Power Series

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Functions Represented by Power Series

Definition: [Functions Represented by a Power Series]

Let \( \sum_{n=0}^{\infty} a_n (x - a)^n \) be a power series with radius of convergence \( R > 0 \). Let \( I \) be the interval of convergence for \( \sum_{n=0}^{\infty} a_n (x - a)^n \).

Let \( f \) be the function defined on the interval \( I \) by the formula

\[
f(x_0) = \sum_{n=0}^{\infty} a_n (x_0 - a)^n \tag{*}
\]

for each \( x_0 \in I \).

We say that the function \( f \) is represented by the power series \( \sum_{n=0}^{\infty} a_n (x - a)^n \) on \( I \).
Functions Represented by Power Series

**Question:** Suppose that $f$ and $g$ are represented by power series centered at $x = a$ of the form

$$f(x) = \sum_{n=0}^{\infty} a_n(x - a)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n(x - a)^n$$

with intervals of convergence $I_f$ and $I_g$ respectively.

Can this information be used to build a power series representation for $f + g$?

**Observation:** If $x_0 \in I_f \cap I_g$, then

$$(f + g)(x_0) = f(x_0) + g(x_0)$$

$$= \lim_{k \to \infty} \sum_{n=0}^{k} a_n(x_0 - a)^n + \lim_{k \to \infty} \sum_{n=0}^{k} b_n(x_0 - a)^n$$

$$= \lim_{k \to \infty} \sum_{n=0}^{k} (a_n + b_n)(x_0 - a)^n$$

$$= \sum_{n=0}^{\infty} (a_n + b_n)(x_0 - a)^n$$
Theorem: [Addition of Power Series]

Assume that \( f \) and \( g \) are represented by power series centered at \( x = a \) with

\[
f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n
\]

and

\[
g(x) = \sum_{n=0}^{\infty} b_n (x - a)^n,
\]

respectively.

Assume also that the radii of convergence of these series are \( R_f \) and \( R_g \) with intervals of convergence \( I_f \) and \( I_g \). Then if \( x \in I_f \cap I_g \),

\[
(f + g)(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x - a)^n.
\]

Moreover, if \( R_f \neq R_g \), then the radius of convergence of the power series representing \( f + g \) is \( R = \min\{R_f, R_g\} \) and the interval of convergence is \( I = I_f \cap I_g \).

If \( R_f = R_g \), then \( R \geq R_f \).
**Multiplication by** \((x - a)^m\)

**Remark:** Assume that \(h(x) = (x - a)^m f(x)\) where \(m \in \mathbb{N}\). We might guess that \(h\) would be represented by the following power series centered at \(x = a\):

\[
h(x) = (x - a)^m \sum_{n=0}^{\infty} a_n(x - a)^n = \sum_{n=0}^{\infty} a_n(x - a)^{n+m}.
\]

**Observation:** If \(x_0 \in I_f\), then

\[
h(x_0) = (x_0 - a)^m f(x_0) = (x_0 - a)^m \lim_{k \to \infty} \sum_{n=0}^{k} a_n(x_0 - a)^n
\]

\[
= \lim_{k \to \infty} \sum_{n=0}^{k} a_n(x_0 - a)^{n+m}
\]

\[
= \sum_{n=0}^{\infty} a_n(x_0 - a)^{n+m}
\]
Theorem: [Multiplication of Power Series by \((x - a)^m\)]

Assume that \(f\) is represented by a power series centered at \(x = a\) as

\[
f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n
\]

with radius of convergence \(R_f\) and interval of convergence \(I_f\).
Assume that \(h(x) = (x - a)^m f(x)\) where \(m \in \mathbb{N}\). Then \(h\) can also be represented by a power series centered at \(x = a\) with

\[
h(x) = \sum_{n=0}^{\infty} a_n (x - a)^{n+m}
\]

Moreover, the series that represents \(h\) has the same radius of convergence and the same interval of convergence as the series that represents \(f\).
**Multiplication by \((x - a)^m\)**

**Example:** Find a power series representation centered at \(x = 0\) for

\[
h(x) = \frac{x}{1 - x}.
\]

**Note:** We have that

\[
h(x) = x \cdot \frac{1}{1 - x}
\]

and

\[
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n
\]

for all \(x \in (-1, 1)\).

Hence

\[
h(x) = \frac{x}{1 - x} = x \cdot \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}
\]

for all \(x \in (-1, 1)\).
Composition with $c \cdot x^m$

**Question:** Assume that $f$ has a power series representation

$$f(u) = \sum_{n=0}^{\infty} a_n u^n$$

centered at $u = 0$ with interval of convergence $I_f$. Can we find a power series representation for

$$h(x) = f(c \cdot x^m)$$

centered at $x = 0$ where $c$ is some non-zero constant?

**Observation:** If we choose $x_0$ so that $c \cdot x_0^m \in I_f$, then substituting $c \cdot x_0^m$ for $u$ gives us

$$h(x_0) = f(c \cdot x_0^m) = \sum_{n=0}^{\infty} a_n (c \cdot x_0^m)^n = \sum_{n=0}^{\infty} (a_n \cdot c^n) x_0^{mn}.$$
Theorem: [Composition with $c \cdot x^m$]

Assume that $f$ has a power series representation

$$f(u) = \sum_{n=0}^{\infty} a_n u^n$$

centered at $u = 0$ with radius of convergence $R_f$ and interval of convergence $I_f$. Let $h(x) = f(c \cdot x^m)$ where $c$ is a non-zero constant. Then $h$ has a power series representation centered at $x = 0$ of the form

$$h(x) = f(c \cdot x^m) = \sum_{n=0}^{\infty} (a_n \cdot c^n) x^{mn}$$

The interval of convergence is

$$I_h = \{ x \in \mathbb{R} \mid c \cdot x^m \in I_f \}$$

and the radius of convergence is $R_h = \frac{m}{|c|} \sqrt{R_f}$ if $R_f < \infty$ and $R_h = \infty$ otherwise.
Example

**Question:** Find a power series representation for \( f(x) = \frac{x}{1 - 2x^2} \) centered at \( x = 0 \).

**Solution:** We know that

\[
\frac{1}{1 - u} = \sum_{n=0}^{\infty} u^n
\]

for \( u \in (-1, 1) \). Then

\[
\frac{1}{1 - 2x^2} = \sum_{n=0}^{\infty} (2x^2)^n = \sum_{n=0}^{\infty} 2^n x^{2n}
\]

provided that

\[
2x^2 \in (-1, 1) \Rightarrow x^2 \in \left(-\frac{1}{2}, \frac{1}{2}\right).
\]

Therefore,

\[
\frac{x}{1 - 2x^2} = x \cdot \sum_{n=0}^{\infty} 2^n x^{2n} = \sum_{n=0}^{\infty} 2^n x^{2n+1}
\]

if and only if \( x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \).