Binomial Series

Created by

Barbara Forrest and Brian Forrest
Theorem: [Binomial Theorem]

Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then for each $x \in \mathbb{R}$ we have that

$$(a + x)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} x^k$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

In particular, when $a = 1$ we have

$$(1 + x)^n = 1 + \sum_{k=1}^{n} \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} x^k$$
Observation: Consider the expression
\[
\frac{n(n - 1)(n - 2) \cdots (n - k + 1)}{k!}
\]
If \(k > n\), then one of the terms in the expression
\[
n(n - 1)(n - 2) \cdots (n - k + 1)
\]
will be 0 and hence we would have
\[
\frac{n(n - 1)(n - 2) \cdots (n - k + 1)}{k!} = 0
\]
Consequently, we also have
\[
(1 + x)^n = 1 + \sum_{k=1}^{\infty} \frac{n(n - 1)(n - 2) \cdots (n - k + 1)}{k!} x^k
\]
Hence the polynomial function \((1 + x)^n\) is actually represented by the power series
\[
1 + \sum_{k=1}^{\infty} \frac{n(n - 1)(n - 2) \cdots (n - k + 1)}{k!} x^k
\]
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**Question:** Suppose that \( \alpha \in \mathbb{R} \). Is there an analog of the Binomial Theorem for the function

\[
(1 + x)^\alpha
\]

**Definition: [Generalized Binomial Coefficients and Binomial Series]**

Let \( \alpha \in \mathbb{R} \) and let \( k \in \{0, 1, 2, 3, \ldots\} \). Then we define the generalized binomial coefficient \( \alpha \) choose \( k \) by

\[
\binom{\alpha}{k} = \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!}
\]

if \( k \neq 0 \) and

\[
\binom{\alpha}{0} = 1
\]

We also define the generalized binomial series for \( \alpha \) to be the power series

\[
1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k
\]
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Remark: If \( b_k = | \binom{\alpha}{k} | \). Then for \( k \geq 1 \)

\[
\frac{b_{k+1}}{b_k} = \frac{|\alpha - k|}{k + 1}
\]

Hence

\[
\lim_{k \to \infty} \frac{b_{k+1}}{b_k} = \lim_{k \to \infty} \frac{|\alpha - k|}{k + 1} = 1
\]

Therefore, the radius of convergence for the binomial series is 1. In particular, the series converges absolutely on \((-1, 1)\).

Remark: Is

\[
(1 + x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k
\]
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Observation 1:

If $k \geq 1$

\[
\binom{\alpha}{k+1}(k+1) + \binom{\alpha}{k}k = \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k!}(\alpha - k) + \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k!}(k) = \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k!}(\alpha) = \alpha \binom{\alpha}{k}
\]
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**Observation 2:** Next we let

\[ f(x) = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \]

for each \( x \in (-1, 1) \). We claim that

\[ f'(x) + xf'(x) = \alpha f(x) \]

for each \( x \in (-1, 1) \). Term-by-term differentiation gives us that

\[
\begin{align*}
  f'(x) + xf'(x) &= \sum_{k=1}^{\infty} \binom{\alpha}{k} k x^{k-1} + \sum_{k=1}^{\infty} \binom{\alpha}{k} k x^k \\
  &= \binom{\alpha}{1} + \sum_{k=2}^{\infty} \binom{\alpha}{k} k x^{k-1} + \sum_{k=1}^{\infty} \binom{\alpha}{k} k x^k \\
  &= \alpha + \sum_{k=1}^{\infty} \binom{\alpha}{k+1} (k+1) x^{k} + \sum_{k=1}^{\infty} \binom{\alpha}{k} k x^k \\
  &= \alpha + \sum_{k=1}^{\infty} \binom{\alpha}{k+1} (k+1) + \binom{\alpha}{k} x^k
\end{align*}
\]
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Observation 2 (continued):

If $k \geq 1$ we have

$$
\binom{\alpha}{k+1}(k+1) + \binom{\alpha}{k}k = \alpha \binom{\alpha}{k}
$$

It follows that

$$
f'(x) + xf'(x) = \alpha + \alpha \sum_{k=1}^{\infty} \binom{\alpha}{k} x^k
$$

$$
= \alpha \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k
$$

$$
= \alpha f(x)
$$

as claimed.
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Observation 3: If

\[ g(x) = \frac{f(x)}{(1 + x)^\alpha}, \]

then \( g \) is differentiable on \((-1, 1)\) and since \( \alpha f(x) = (1 + x) f'(x) \),

\[
g'(x) = \frac{f'(x)(1 + x)^\alpha - \alpha f(x)(1 + x)^{\alpha - 1}}{(1 + x)^{2\alpha}}
\]

\[
= \frac{f'(x)(1 + x)^\alpha - (1 + x)f'(x)(1 + x)^{\alpha - 1}}{(1 + x)^{2\alpha}}
\]

\[
= \frac{f'(x)(1 + x)^\alpha - f'(x)(1 + x)^\alpha}{(1 + x)^{2\alpha}}
\]

\[
= 0
\]

Since \( g'(x) = 0 \) for all \( x \in (-1, 1) \), \( g(x) \) is constant on this interval. However,

\[ g(0) = f(0) = 1. \]

Therefore, \( g(x) = 1 \) for all \( x \in (-1, 1) \). It follows that

\[ f(x) = (1 + x)^\alpha \]
Theorem: [Generalized Binomial Theorem]

Let $\alpha \in \mathbb{R}$. Then for each $x \in (-1, 1)$ we have that

$$(1 + x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$
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**Example:** Use the Generalized Binomial Theorem to find a power series representation for \((1 + x)^{-2}\).

**Note:** By the Generalized Binomial Theorem we have

\[
(1 + x)^{-2} = \sum_{k=0}^{\infty} \left(\begin{array}{c}-2 \\ k\end{array}\right) x^k
\]

We have that for \(k \geq 1\)

\[
\left(\begin{array}{c}-2 \\ k\end{array}\right) = \frac{(-2)(-2 - 1) \cdots (-2 - k + 1)}{k!} = (-1)^k (k + 1)
\]

and

\[
\left(\begin{array}{c}-2 \\ 0\end{array}\right) = 1 = (-1)^0 (0 + 1)
\]

Therefore,

\[
(1 + x)^{-2} = \sum_{k=0}^{\infty} (-1)^k (k + 1) x^k
\]

\[
= \sum_{k=1}^{\infty} (-1)^{k-1} k x^{k-1}
\]