Monotone Convergence Theorem

Created by

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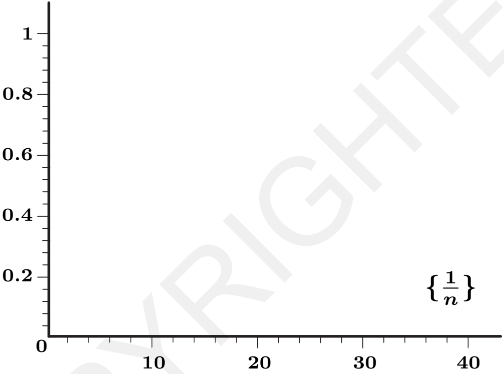
Monotonic Sequences

Definition: [Monotonic Sequences]

We say that a sequence $\{a_n\}$ is:

- *increasing* if $a_n < a_{n+1}$, for all $n \in \mathbb{N}$.
- *non-decreasing* if $a_n \leq a_{n+1}$, for all $n \in \mathbb{N}$.
- *decreasing* if $a_n > a_{n+1}$, for all $n \in \mathbb{N}$.
- *non-increasing* if $a_n \geq a_{n+1}$, for all $n \in \mathbb{N}$. e
- *monotonic* if $\{a_n\}$ is either non-decreasing or non-increasing.

Examples of Monotonic Sequences



- The sequence $\{\frac{1}{n}\}$ is decreasing.

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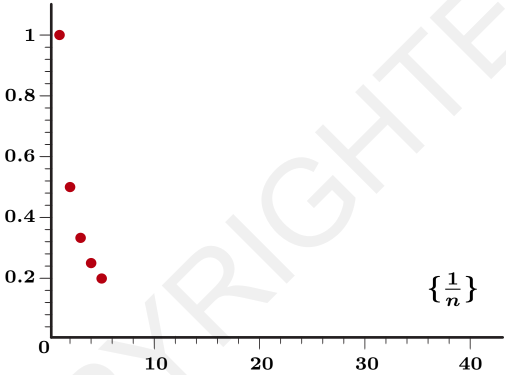

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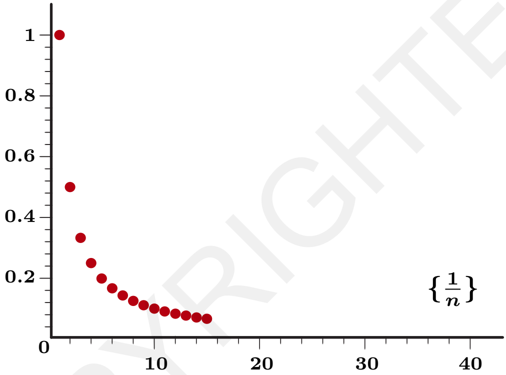

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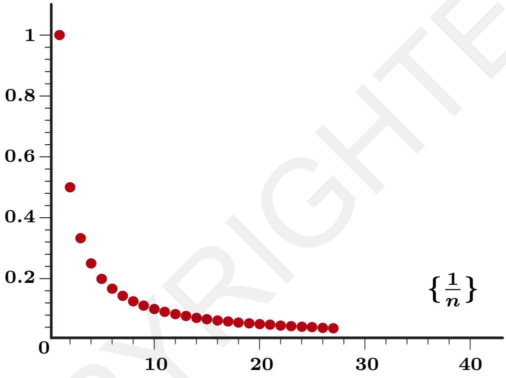

- The sequence $\left\{\frac{1}{n}\right\}$ is decreasing.

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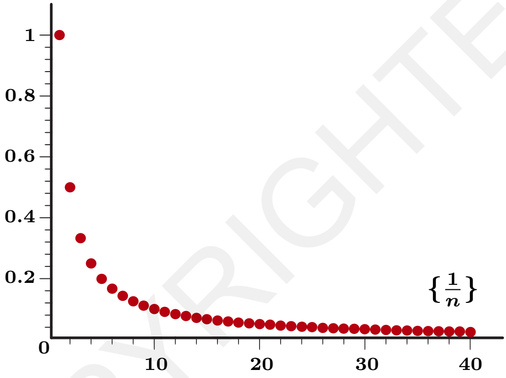

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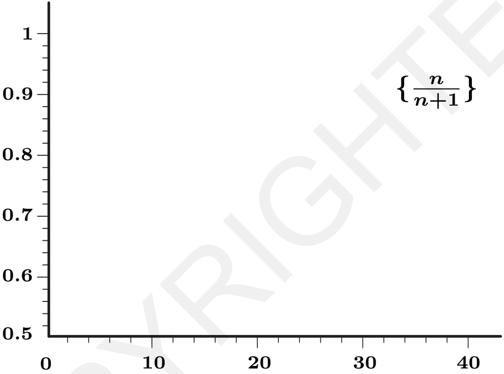

- The sequence $\{\frac{1}{n}\}$ is decreasing.
- The sequence $\{\frac{n}{n+1}\} = \{1 - \frac{1}{n+1}\}$ is increasing.

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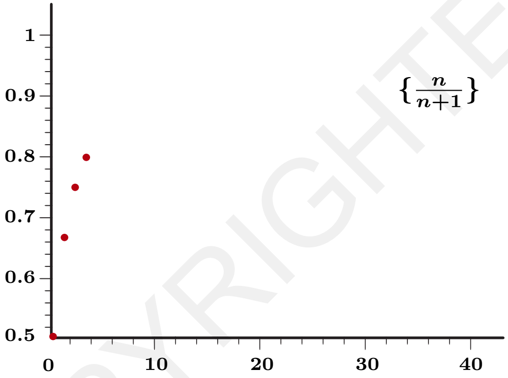

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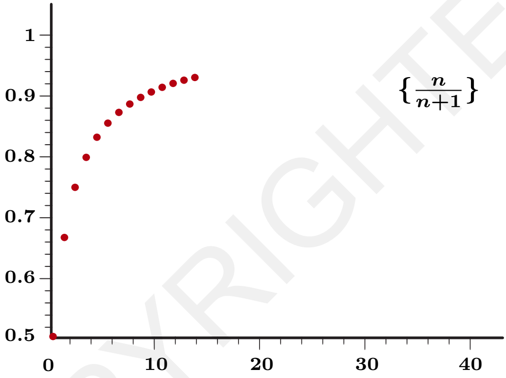

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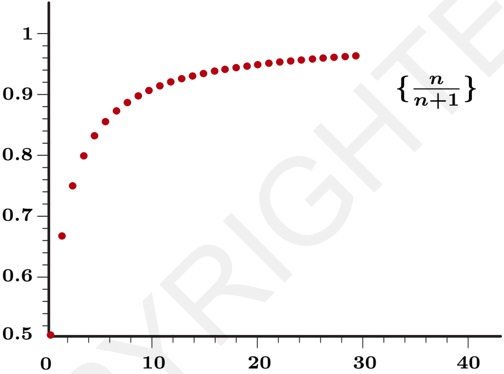

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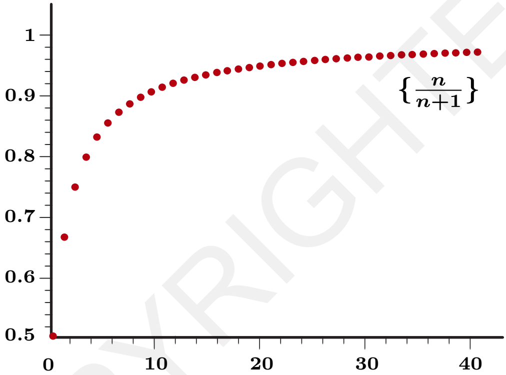

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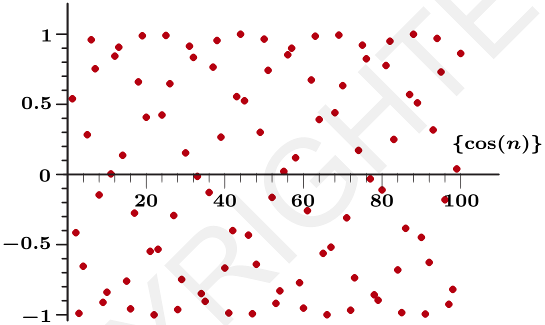

- The sequence $\{\frac{1}{n}\}$ is decreasing.
- The sequence $\{\frac{n}{n+1}\} = \{1 - \frac{1}{n+1}\}$ is increasing.
- The sequence $\{cos(n)\}$ is neither non-decreasing or non-increasing.

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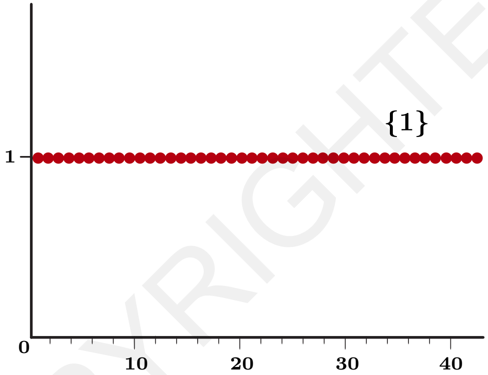


- The sequence $\{\frac{1}{n}\}$ is decreasing.
- The sequence $\{\frac{n}{n+1}\} = \{1 - \frac{1}{n+1}\}$ is increasing.
- The sequence $\{cos(n)\}$ is neither non-decreasing or non-increasing.
- The constant sequence $\{1\}$ is both non-decreasing and non-increasing.

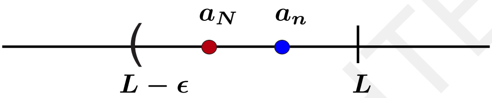
Monotone Convergence Theorem

Theorem: [Monotone Convergence Theorem (MCT)]

1) If $\{a_n\}$ is non-decreasing and bounded above, then $\{a_n\}$ converges to $L = lub\{a_n\}$.
2) If $\{a_n\}$ is non-decreasing and unbounded, then $\{a_n\}$ diverges to ∞.

Note: A non-decreasing sequence converges if and only if it is bounded.

Monotone Convergence Theorem



Proof of (1):

Assume that $\{a_n\}$ is non-decreasing and bounded with $L = lub\{a_n\}$. Let $\epsilon > 0$. Then $L - \epsilon < L$, so $L - \epsilon$ is **not** an upper bound of $\{a_n\}$. Hence, we can find an $N \in \mathbb{N}$ such that $L - \epsilon < a_N$.

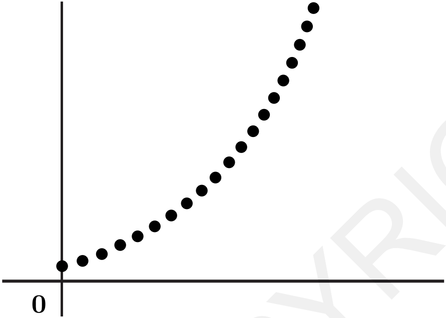
If $n \geq N$, then

$$L - \epsilon < a_N \leq a_n \leq L.$$

This shows that if $n \geq N$, then $| a_n - L | < \epsilon$. So

$$\lim_{n \to \infty} a_n = L.$$

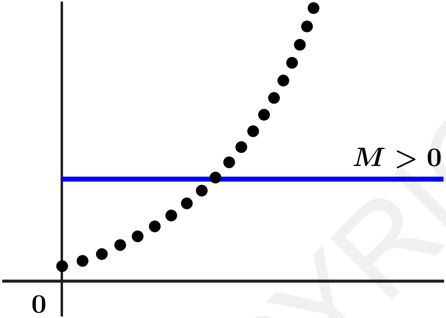
Monotone Convergence Theorem



Proof of (2):

Assume that $\{a_n\}$ is non-decreasing and unbounded.

Monotone Convergence Theorem

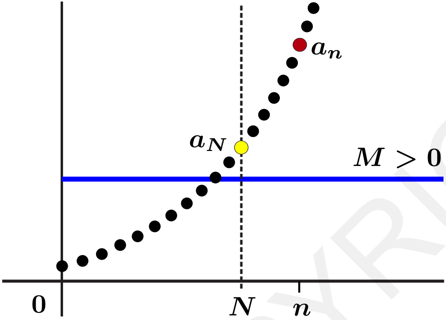


Proof of (2):

Assume that $\{a_n\}$ is non-decreasing and unbounded.

Let $M > 0$.

Monotone Convergence Theorem



Proof of (2):

Assume that $\{a_n\}$ is non-decreasing and unbounded.

Let $M > 0$.

Since $\{a_n\}$ is unbounded there exists $N \in \mathbb{N}$ such that

$$M < a_N.$$

Since $\{a_n\}$ is non-decreasing, if $n \geq N$ then

$$M < a_N \leq a_n.$$

So $\{a_n\}$ diverges to ∞.

Monotone Convergence Theorem

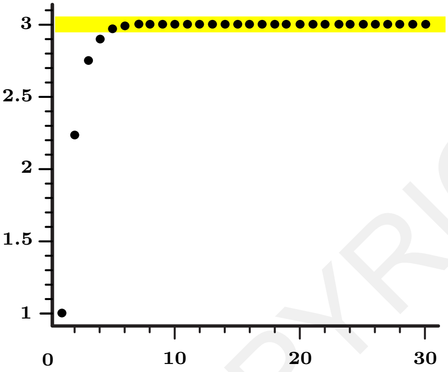
Note: We can also show that:

- if $\{a_n\}$ is non-increasing and bounded below, then

$$\lim_{n\to\infty} a_n = glb\{a_n\}.$$

- if $\{a_n\}$ is non-increasing and unbounded, it diverges to $-\infty$.

Example



Example: Let $\{a_n\}$ be defined recursively by

$$a_1 = 1, \; a_{n+1} = \sqrt{3 + 2a_n}.$$

Show that $\{a_n\}$ converges.

Claim:

$$0 \leq a_n \leq a_{n+1} \leq 3.$$

Example

Proof of the Claim:

Let $P(n)$ be the statement that

$$0 \leq a_n \leq a_{n+1} \leq 3.$$

Step 1: Show $P(1)$ holds.

We have

$$a_2 = \sqrt{3 + 2 \cdot 1} = \sqrt{5}$$

so

$$0 \leq a_1 = 1 \leq \sqrt{5} = a_2 \leq 3.$$

$\Longrightarrow P(1)$ holds.

Example

Step 2: Assume $P(k)$ holds and then show that $P(k+1)$ holds:

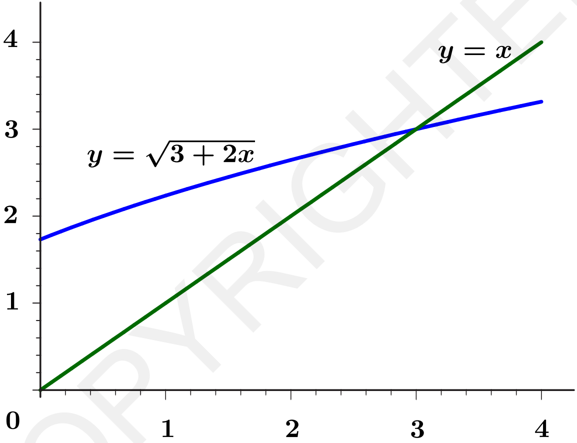
$$\begin{aligned}
P(k) &\implies 0 \le a_k \le a_{k+1} \le 3 \\
&\implies 0 \le 2a_k \le 2a_{k+1} \le 6 \\
&\implies 3 \le 3 + 2a_k \le 3 + 2a_{k+1} \le 9 \\
&\implies \sqrt{3} \le \sqrt{3 + 2a_k} \le \sqrt{3 + 2a_{k+1}} \le \sqrt{9} \\
&\implies 0 \le a_{k+1} \le a_{k+2} \le 3
\end{aligned}$$

$\implies P(k+1)$ holds.

Conclusion: $\{a_n\}$ is non-decreasing and bounded above by 3. By the MCT $\{a_n\}$ converges.

Question: Does this prove $\lim_{n\to\infty} a_n = 3$?

Example



$$a_1 = 1 \qquad a_{n+1} = \sqrt{3 + 2a_n}.$$

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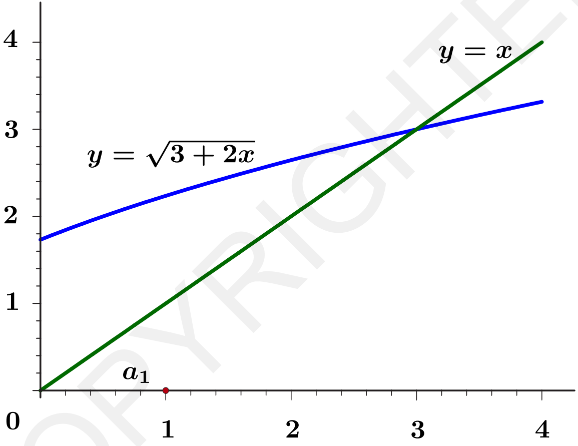

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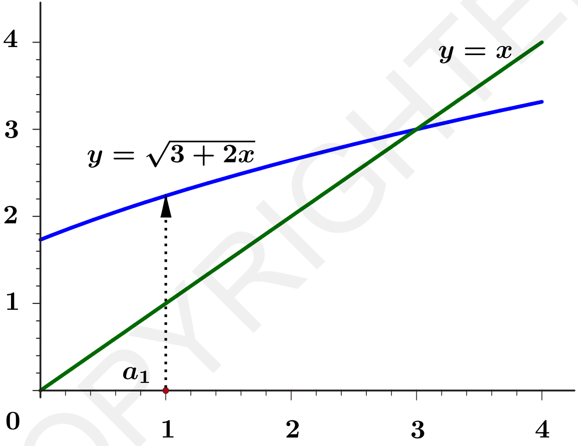

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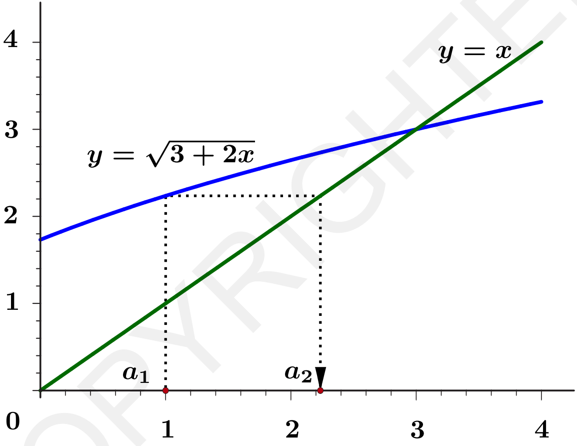

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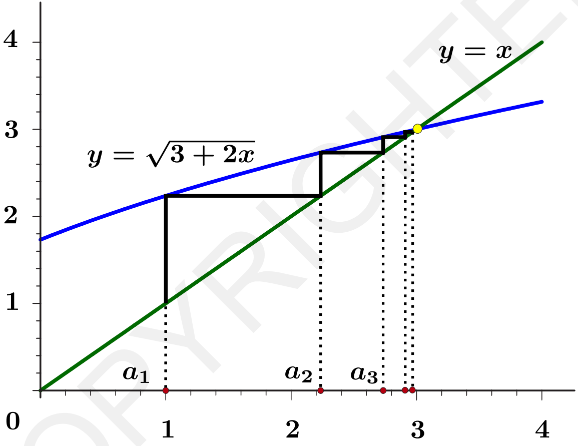

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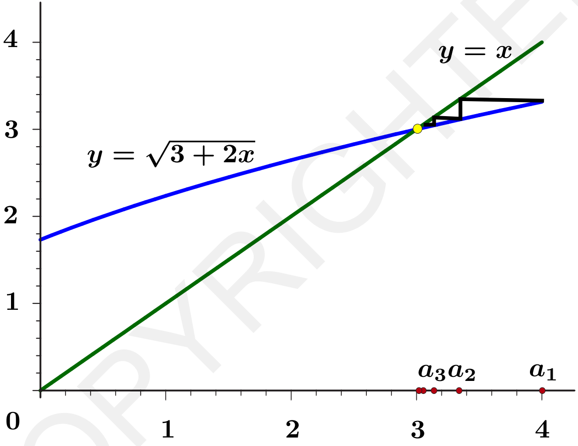
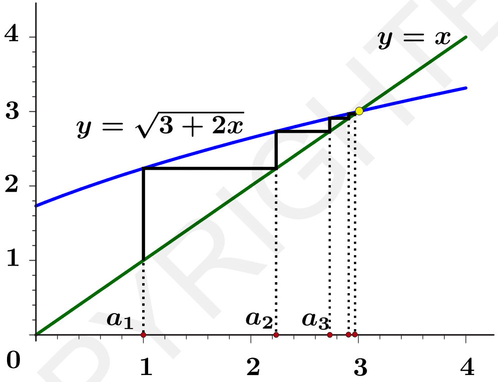


$$a_1 = 4 \qquad a_{n+1} = \sqrt{3 + 2a_n}.$$

Example



Question: Why 3?

Graphically: The graphs of $y = x$ and $y = \sqrt{3 + 2x}$ intersect at $x = 3$.

Example

Algebraically: Assume $\lim_{n\to\infty} a_n = L$.

$$\begin{aligned} a_n \to L \quad &\Rightarrow \quad 3 + 2a_n \to 3 + 2L \\ &\Rightarrow \quad \sqrt{3 + 2a_n} \to \sqrt{3 + 2L} \\ &\Rightarrow \quad a_{n+1} \to \sqrt{3 + 2L}. \end{aligned}$$

Then

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \sqrt{3 + 2L}$$

$$\Rightarrow L = \sqrt{3 + 2L}.$$

Example

If

$$\begin{aligned} L = \sqrt{3 + 2L} \quad &\Rightarrow \quad L^2 = 3 + 2L \\ &\Rightarrow \quad L^2 - 2L - 3 = 0 \\ &\Rightarrow \quad (L - 3)(L + 1) = 0 \end{aligned}$$

then

$$L = 3 \quad \text{or} \quad L = -1.$$

Since $a_n > 0 \Rightarrow L = 3$.

Summary

Summary:

- If $\{a_n\}$ is non-decreasing and bounded above $\Rightarrow \lim_{n\to\infty} a_n = lub\{a_n\}$.
- If $\{a_n\}$ is non-decreasing and unbounded $\Rightarrow \lim_{n\to\infty} a_n = \infty$.
- If $\{a_n\}$ is non-increasing and bounded below $\Rightarrow \lim_{n\to\infty} a_n = glb\{a_n\}$.
- If $\{a_n\}$ is non-increasing and unbounded $\Rightarrow \lim_{n\to\infty} a_n = -\infty$.