The Limit Comparison Test

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Failure of the Comparison Test

**Question:** Does the series
\[ \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \]
converge?

We know that
\[ 0 < \sin \left( \frac{1}{n} \right) \leq \frac{1}{n} \]
for all \( n \in \mathbb{N} \) but \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges so the Comparison Test fails.

Since \( \lim_{n \to \infty} \frac{1}{n} = 0 \), the Fundamental Trig Limits shows that
\[ \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n} \right)}{\frac{1}{n}} = 1 \]
so for large \( n \) we have
\[ \sin \left( \frac{1}{n} \right) \approx \frac{1}{n} \]

Does this mean that
\[ \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \]
also diverges?
The Limit Comparison Test

**Theorem: [The Limit Comparison Test for Series]**

Assume that $a_n > 0$ and $b_n > 0$ for each $n \in \mathbb{N}$. Assume also that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L$$

where either $L \in \mathbb{R}$ or $L = \infty$.

1) If $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

2) If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   Equivalently, if $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$.

3) If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

   Equivalently, if $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$. 
The Limit Comparison Test

Proof of the Limit Comparison Test: First we assume that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L$$

1) If $0 < L < \infty$, the interval $(\frac{L}{2}, 2L)$ is an open interval containing $L$. It follows that we can find a cutoff $N \in \mathbb{N}$ so that if $n \geq N$, then

$$\frac{L}{2} < \frac{a_n}{b_n} < 2L$$

or equivalently that

$$\frac{L}{2} \cdot b_n < a_n < 2Lb_n$$

If $\sum_{n=1}^{\infty} a_n$ converges, then the Comparison Test shows that

$$\sum_{n=1}^{\infty} \frac{L}{2} \cdot b_n$$

converges and hence so does

$$\sum_{n=1}^{\infty} b_n$$
Proof of the Limit Comparison Test (continued):

1) Since

\[ \frac{L}{2} \cdot b_n < a_n < 2Lb_n, \]

if \( \sum_{n=1}^{\infty} b_n \) converges, then so does

\[ \sum_{n=1}^{\infty} 2L \cdot b_n \]

By the Comparison Test

\[ \sum_{n=1}^{\infty} a_n \]

also converges.
The Limit Comparison Test

Proof of the Limit Comparison Test (continued):

2) If $L = 0$, then we can find a cut off $N \in \mathbb{N}$ so that if $n \geq N$, then

\[
0 < \frac{a_n}{b_n} < 1
\]

or equivalently that

\[
0 < a_n < b_n
\]

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well by the Comparison Test.

Equivalently, if $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. 
Proof of the Limit Comparison Test (continued):

3) If $L = \infty$, then we can find a cut off $N \in \mathbb{N}$ so that if $n \geq N$, then

$$\frac{a_n}{b_n} > 1$$

or equivalently that

$$b_n < a_n$$

If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges as well by the Comparison Test.

Equivalently, if $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$. 
The Limit Comparison Test

Summary:

1) If \( \lim_{n \to \infty} \frac{a_n}{b_n} = L \) where \( 0 < L < \infty \), then for large \( n \) we have

\[
\frac{a_n}{b_n} \approx L
\]

or

\[
a_n \approx Lb_n.
\]

When \( \lim_{n \to \infty} \frac{a_n}{b_n} = L \) where \( 0 < L < \infty \), we say that \( a_n \) and \( b_n \) have the same order of magnitude. We write

\[
a_n \approx b_n
\]

The Limit Comparison Test says that two positive series with terms of the same order of magnitude will have the same convergence properties.
The Limit Comparison Test

Summary:

2) If \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \), then \( b_n \) must eventually be much larger than \( a_n \).

In this case, we write

\[ a_n \ll b_n \]

and we say that the order of magnitude of \( a_n \) is smaller than the order of magnitude of \( b_n \).

If the smaller series \( \sum_{n=1}^{\infty} a_n \) diverges to \( \infty \), it would make sense that \( \sum_{n=1}^{\infty} b_n \) also diverges to \( \infty \).

3) If \( \lim_{n \to \infty} \frac{a_n}{b_n} = \infty \), then \( a_n \) must eventually be much larger than \( b_n \).

That is \( b_n \ll a_n \). This time, if the larger series \( \sum_{n=1}^{\infty} a_n \) converges, it would make sense that \( \sum_{n=1}^{\infty} b_n \) would converge as well.
Example: The series

$$\sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right)$$

diverges.

Since

$$\lim_{n \to \infty} \frac{\sin \left( \frac{1}{n} \right)}{\frac{1}{n}} = 1$$

and since

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges the Limit Comparison Test shows that

$$\sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right)$$

also diverges.
Example: The series \( \sum_{n=1}^{\infty} \frac{2n}{n^3-n+1} \) converges.

Let \( a_n = \frac{2n}{n^3-n+1} \) and \( b_n = \frac{1}{n^2} \). Then

\[
\frac{a_n}{b_n} = \frac{\frac{2n}{n^3-n+1}}{\frac{1}{n^2}} = \frac{2n^3}{n^3 - n + 1}
\]

\[
= \frac{n^3 \left( 1 - \frac{1}{n^2} + \frac{1}{n^3} \right)}{1 - \frac{1}{n^2} + \frac{1}{n^3}}
\]

Therefore,

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{2}{1 - \frac{1}{n^2} + \frac{1}{n^3}} = \frac{2}{1} = 2.
\]

Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, the Limit Comparison Test shows that

\[
\sum_{n=1}^{\infty} \frac{2n}{n^3-n+1}
\]

converges.