Introduction to Series

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A Problem to Consider

Problem:

Can infinitely many tasks be performed in a finite amount of time?
The Paradox of Achilles

Achilles reaches the next point, and again the tortoise has moved ahead.

And so on . . .

**Conclusion:** Achilles can never catch the tortoise!!!
Resolving The Paradox of Achilles

We call each time Achilles moves to where the tortoise was a *stage*.

$\begin{align*}
&d_1 = \text{distance Achilles traveled in stage 1} \Rightarrow t_1 = \text{time to complete stage 1} \\
&d_2 = \text{distance Achilles traveled in stage 2} \Rightarrow t_2 = \text{time to complete stage 2} \\
&d_n = \text{distance Achilles traveled in stage } n \Rightarrow t_n = \text{time to complete stage } n \\
\end{align*}$

Time to catch the Tortoise $= t_1 + t_2 + \cdots + t_n + \cdots = \infty \, ?$
Introduction to Series

Problem:
Can we add infinitely many numbers at the same time?

More precisely, given a sequence \( \{a_n\} \), we can form the formal sum

\[
a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n
\]

which is called a series?

Question:
What does this formal sum represent? Does it have a value?
Example: What is

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots
\]

\[
= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots?
\]
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Geometric Interpretation
Introduction to Series

Geometric Interpretation

\[ \frac{1}{2} + \]
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Geometric Interpretation

\[ \frac{1}{2} + \frac{1}{4} + \cdots \]
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Geometric Interpretation

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots
\]
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Geometric Interpretation

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots
\]
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Geometric Interpretation

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots
\]
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Geometric Interpretation

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots
\]
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Geometric Interpretation

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots
\]
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Geometric Interpretation

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \cdots = 1 \]
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Geometric Interpretation

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \cdots \]
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Geometric Interpretation

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \cdots
\]
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Geometric Interpretation

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \cdots \]
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Geometric Interpretation

\[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \cdots\]

\[= 1?\]
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Sum of Areas: $\frac{1}{2}$

Total Area Covered: $1 - \frac{1}{2}$
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Sum of Areas

\[ \frac{1}{2} + \frac{1}{4} \]

Total Area Covered

\[ 1 - \frac{1}{4} \]
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Sum of Areas

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \]

Total Area Covered

\[ 1 - \frac{1}{8} \]
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Sum of Areas
\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \]

Total Area Covered
\[ 1 - \frac{1}{16} \]
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Sum of Areas

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \]

Total Area Covered

\[ 1 - \frac{1}{32} \]
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\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} \]

Sum of Areas

Total Area Covered

\[ 1 - \frac{1}{64} \]
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Sum of Areas

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}
\]

Total Area Covered

\[
1 - \frac{1}{128}
\]
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Sum of Areas

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256}
\]

Total Area Covered

\[
1 - \frac{1}{256}
\]
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Sum of Areas

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512}
\]

Total Area Covered

\[
1 - \frac{1}{512}
\]
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Sum of Areas

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024}
\]

Total Area Covered

\[
1 - \frac{1}{1024}
\]
Introduction to Series

Sum of Areas

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \cdots + \frac{1}{2^k}
\]

Total Area Covered

\[
1 - \frac{1}{2^k}
\]


\[
\lim_{k \to \infty} \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} = \lim_{k \to \infty} \sum_{n=1}^{k} \frac{1}{2^n} = \lim_{k \to \infty} 1 - \frac{1}{2^k} = 1
\]

Note:
Definition: [Series]

Given a sequence \( \{a_n\} \), the formal sum

\[
a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots
\]

is called a series. (The series is called formal because we have not yet given it a meaning numerically.)

The \( a_n \)'s are called the terms of the series. For each term \( a_n \), the number \( n \) is called the index of the term.

We denote the series by

\[
\sum_{n=1}^{\infty} a_n.
\]
Convergent/Divergent Series

Definition: [Convergent Series]

Given a sequence \( \{a_n\} = \{a_1, a_2, a_3, \ldots\} \), we define the \( k \text{th} \) partial sum \( S_k \) of the series \( \sum_{n=1}^{\infty} a_n \) by

\[
S_k = a_1 + a_2 + \cdots + a_k = \sum_{n=1}^{k} a_n.
\]

We say that the series \( \sum_{n=1}^{\infty} a_n \) converges if the sequence of partial sums \( \{S_k\} \) converges. In this case, we write

\[
\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} S_k
\]

Otherwise, we say that the series diverges and the sum has no defined value.
Example:

Suppose $a_n = \frac{1}{2^n}$. We know that

$$S_k = \sum_{n=1}^{k} \frac{1}{2^n} = 1 - \frac{1}{2^k} \to 1.$$ 

Hence, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges with

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$
Why Use Limits?

**Question:** Why use limits?

Suppose \( a_n = (-1)^{n+1} \). Then the formal sum looks like

\[
a_1 + a_2 + a_3 + \cdots = 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots
\]

We could parenthesize the formal sum as follows:

\[
[1 + (-1)] + [1 + (-1)] + [1 + (-1)] + \cdots = 0 + 0 + 0 + \cdots = 0.
\]

Alternatively, we could parenthesize the formal sum as:

\[
1 + [(-1)+1] + [(-1)+1] + [(-1)+1] + \cdots = 1 + 0 + 0 + 0 + \cdots = 1.
\]

Our result is ambiguous; the “sum” changes if we change the way we parenthesize the terms!
Why Use Limits?

Observe:

\[
S_1 = 1 \\
S_2 = 1 - 1 = 0 \\
S_3 = 1 - 1 + 1 = 1 \\
S_4 = 1 - 1 + 1 - 1 = 0
\]

We get

\[
S_k = 1 + (-1) + 1 + \cdots + (-1)^{k+1} = \begin{cases} 
1 & \text{if } k \text{ is odd}, \\
0 & \text{if } k \text{ is even}.
\end{cases}
\]

Thus, \( \{S_k\} = \{1, 0, 1, 0, 1, 0, \cdots \} \) diverges.
Example: Determine if the series

\[ \sum_{n=1}^{\infty} \frac{1}{n^2 + n} \]

converges or diverges.

Solution: Observe that

\[ a_n = \frac{1}{n^2 + n} = \frac{1}{n(n + 1)} = \frac{1}{n} - \frac{1}{n + 1} \]

so the series becomes

\[ \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n + 1} \right). \]

We have

\[ S_k = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{k} - \frac{1}{k + 1}) \]

\[ = 1 - (\frac{1}{2} - \frac{1}{2}) - (\frac{1}{3} - \frac{1}{3}) - (\frac{1}{4} - \frac{1}{4}) - \cdots - (\frac{1}{k} - \frac{1}{k}) - \frac{1}{k + 1} \]

\[ = 1 - 0 - 0 - 0 - \cdots - 0 - \frac{1}{k + 1} = 1 - \frac{1}{k + 1} \rightarrow 1 \]